FIBRATIONS AND SHEAVES

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ABSTRACT

The purpose of this book is to give a systematic treatment of fibration theory and sheaf theory, the emphasis being on the foundational essentials.

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STACKS

SO. CATEGORICAL CONVENTIONS

In this book, the foundation for category theory is the "one universe" approach taken by Herrlich-Strecker[†]. The key words are "set", "class", and "conglomerate". Thus the issue is not only one of size but also of membership (every set is a class and every class is a conglomerate).

0.1 DEFINITION A category C is a class of objects Ob C, a class of morphisms Mor C, a function dom:Mor C \rightarrow Ob C, a function cod:Mor C \rightarrow Ob C, and a function $\circ:\{(f,g):f,g\in Mor\ C\ \&\ cod\ f=dom\ g\}\rightarrow Mor\ C\ (\circ(f,g)=g\circ f)$ such that...

0.2 TERMINOLOGY

- Small Category: A category whose morphism class is a set.
- Large Category: A category whose morphism class is a proper class.

[Note: If \underline{C} is a category and if \underline{O} b \underline{C} is a proper class, then \underline{C} is large.]

Given a category \underline{C} and objects $X,Y\in Ob\ \underline{C}$, it is not assumed that the class

$$Mor(X,Y) = \{f: f \in Mor\ \underline{C}, dom\ f = X, cod\ f = Y\}$$

is a set.

- 0.3 DEFINITION A category \underline{C} is said to be <u>locally small</u> if \forall X,Y \in Ob \underline{C} , Mor(X,Y) is a set.
 - 0.4 EXAMPLE SET is a locally small large category.

[†] Category Theory, Heldermann Verlag, 1979; see also Osborne, Basic Homological Algebra, Springer Verlag, 2000.

- 0.5 EXAMPLE TOP is a locally small large category.
- 0.6 EXAMPLE SCH is a locally small large category (cf. 23.20).
- 0.7 REMARK There are abelian categories \underline{A} whose positive derived category $\underline{D}_{\underline{A}}$ is not locally small.
- 0.8 NOTATION <u>CAT</u> is the locally small category whose objects are the small categories and whose morphisms are the functors.

[Note: CAT is a locally small large category.]

0.9 DEFINITION A <u>metacategory</u> \underline{C} is a conglomerate of objects Ob \underline{C} , a conglomerate of morphisms Mor \underline{C} , a function dom:Mor $\underline{C} \rightarrow$ Ob \underline{C} , a function cod:Mor $\underline{C} \rightarrow$ Ob \underline{C} , and a function

 $\circ : \{ (f,g) : f,g \in Mor \ \underline{C} \ \& \ cod \ f = dom \ g \} \ \to Mor \ \underline{C} \ (\circ (f,g) = g \ \circ \ f)$ such that... .

- N.B. Every category is a metacategory.
- 0.10 NOTATION Given categories $\begin{bmatrix} \underline{C} \\ \underline{D} \end{bmatrix}$, the functor category $[\underline{C},\underline{D}]$ is the metacategory whose objects are the functors $F:\underline{C} \to \underline{D}$ and whose morphisms are the natural transformations Nat(F,G) from F to G.
 - 0.11 REMARK Suppose that C and D are nonempty.
- If $F:\underline{C}\to\underline{D}$ is a functor, then $F:Mor\ \underline{C}\to Mor\ \underline{D}$ is a function, i.e., F is a subclass

$$F \subset Mor C \times Mor D.$$

And F is a proper class iff Mor \underline{C} is a proper class.

• If $F,G:\underline{C} \to \underline{D}$ are functors and if $E:F \to G$ is a natural transformation, then $E:Ob \ \underline{C} \to Mor \ \underline{D}$ is a function, i.e., E is a subclass

$$\Xi \subset Ob \ \underline{C} \times Mor \ \underline{D}.$$

And Ξ is a proper class iff Ob C is a proper class.

Accordingly, if Ob \underline{C} is a proper class, then $[\underline{C},\underline{D}]$ is a metacategory, not a category.

[Note: If, however, \underline{C} is small, then $[\underline{C},\underline{D}]$ is a category and if \underline{D} is locally small, then $[\underline{C},\underline{D}]$ is locally small.]

- 0.12 EXAMPLE Let ON be the ordered class of ordinals then ON is a metacategory, not a category.
- 0.13 NOTATION CAT is the metacategory whose objects are the categories and whose morphisms are the functors.

§1. 2-CATEGORIES

It is a question here of establishing notation and reviewing the basics.

1.1 DEFINITION A 2-category $\mathfrak C$ consists of a class O and a function that assigns to each ordered pair $X,Y\in O$ a category $\mathfrak C(X,Y)$ plus functors

$$C_{X,Y,Z}$$
: $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z)$

and

$$I_X:\underline{1} \longrightarrow C(X,X)$$

satisfying the following conditions.

 $(2-cat_1)$ The diagram

commutes.

(2-cat₂) The diagram

commutes.

- 1.2 REMARK It is not assumed that the C(X,Y) are small or even locally small.
- 1.3 TERMINOLOGY Let C be a 2-category.
 - The elements of the class O are called O-cells (denoted X,Y,Z,...).
- The objects of the category $\mathfrak{C}(X,Y)$ are called <u>1-cells</u> (denoted f,g,h,...)

 (and we write $f:X \longrightarrow Y$ or $X \xrightarrow{f} Y$).
- The morphisms of the category $\mathfrak{C}(X,Y)$ are called <u>2-cells</u> (denoted α,β , γ,\ldots) (and we write $\alpha:f$ $\Longrightarrow >g$ or f $\Longrightarrow >g$).
- N.B. It is common practice to define a 2-category by simply delineating the 0-cells, the 1-cells, and the 2-cells, leaving implicit the precise definition of the $\mathfrak{C}(X,Y)$ (as well as the $C_{X,Y,Z}$ and the I_X).
- 1.4 EXAMPLE There is a 2-category 2-REL whose 0-cells are the sets, whose 1-cells $f:X \to Y$ are the subsets f of $X \times Y$, and whose 2-cells $\alpha:f \longrightarrow g$ (f,g $\subset X \times Y$) are defined by stipulating that there is a unique 2-cell from f to g if f $\subset g$ but no 2-cell from f to g otherwise.
- 1.5 EXAMPLE There is a 2-category 2-TOP whose 0-cells are the topological spaces, whose 1-cells are the continuous functions, and whose 2-cells are the homotopy classes of homotopies.
- 1.6 EXAMPLE Let C be a locally small finitely complete category -- then there is a 2-category CAT(C) whose 0-cells are the internal categories in C, whose 1-cells are the internal functors, and whose 2-cells are the internal natural transformations.

 [Note:
 - Take C = SET then the 0-cells in CAT(SET) are the small categories.

• Take $\underline{C} = \underline{CAT}$ -- then the 0-cells in $\underline{CAT}(\underline{CAT})$ are the small double categories.]

1.7 NOTATION

• The composition of

$$f \xrightarrow{\alpha} g \xrightarrow{\beta} h$$

in C(X,Y) is denoted by $\beta \bullet \alpha$.

[Note: Given a 1-cell f, there is a 2-cell $id_f: f \longrightarrow f$ such that $\alpha \bullet id_f = \alpha$ for all $\alpha: f \longrightarrow g$ and $id_f \bullet \beta = \beta$ for all $\beta: h \longrightarrow f$.]

• The image of 1-cells $f:X \to Y$, $k:Y \to Z$ under $C_{X,Y,Z}$ is denoted by $k \circ f$.

[Note: Let l_X be the image of the unique object of \underline{l} under I_X (hence $l_X: X \to X$) — then for any 1-cell $f: X \to Y$,

$$C(l_{X},f) = f \circ l_{X} = f = l_{Y} \circ f = C(f,l_{Y}).$$

• The image of 2-cells f $\xrightarrow{\alpha}$ > g, k $\xrightarrow{\mu}$ > ℓ under $C_{X,Y,Z}$ is denoted by $\mu * \alpha$.

[Note: If α :f \Longrightarrow > g, then

$$\alpha * id_{1_x} = \alpha = id_{1_y} * \alpha$$
.

On the other hand, if $f:X \to Y$, $k:Y \to Z$, then

$$id_k * id_f = id_k \circ f$$
.

To illustrate, suppose given

$$\begin{bmatrix}
f & \frac{\alpha}{\dots} > g & \frac{\beta}{\dots} > h \\
k & \frac{\mu}{\dots} > \ell & \frac{\nu}{\dots} > m.
\end{bmatrix}$$

Then

Therefore

$$(v * \beta) \bullet (\mu * \alpha) = C_{X,Y,Z}(\beta, \nu) \bullet C_{X,Y,Z}(\alpha, \mu)$$

$$= C_{X,Y,Z}((\beta, \nu) \bullet (\alpha, \mu))$$

$$= C_{X,Y,Z}(\beta \bullet \alpha, \nu \bullet \mu)$$

$$= (\nu \bullet \mu) * (\beta \bullet \alpha).$$

1.8 REMARK The equation

$$(v * \beta) \bullet (\mu * \alpha) = (v \bullet \mu) * (\beta \bullet \alpha)$$

is called the exchange principle.

1.9 EXAMPLE Suppose that

Then

$$\mu * \alpha = \begin{bmatrix} (\mu * id_g) \bullet (id_k * \alpha) \\ (id_\ell * \alpha) \bullet (\mu * id_f). \end{bmatrix}$$

1.10 EXAMPLE Suppose that α , β : $1_X \longrightarrow 1_X \longrightarrow$ then

$$\alpha \bullet \beta = \beta \bullet \alpha$$
.

In fact,

$$\alpha \bullet \beta = (id_{1_X} * \alpha) \bullet (\beta * id_{1_X})$$

$$= (id_{1_X} \bullet \beta) * (\alpha \bullet id_{1_X})$$

$$= \beta * \alpha$$

$$= (\beta \bullet id_{1_X}) * (id_{1_X} \bullet \alpha)$$

$$= (\beta * id_{1_X}) \bullet (id_{1_X} * \alpha)$$

$$= \beta \bullet \alpha.$$

1.11 DEFINITION The underlying category UC of a 2-category C has for its class of objects the 0-cells and for its class of morphisms the 1-cells.

[Note: In this context, l_{X} serves as the identity in Mor(X,X).]

1.12 NOTATION Let

$$2-CAT = CAT(SET)$$
 (cf. 1.6).

1.13 EXAMPLE We have

U2-CAT
$$\approx$$
 CAT.

- 1.14 EXAMPLE Every category C determines a 2-category C for which $UC \approx C$. [Let C = C and let C(X,Y) = Mor(X,Y) (viewed as a discrete category).]
- 1.15 DEFINITION Let C be a 2-category -- then a 1-cell $f:X \to Y$ is said to be a 2-isomorphism if there exists a 1-cell $g:Y \to X$ and invertible 2-cells

- 1.16 DEFINITION Let C be a 2-category -- then 0-cells X and Y are said to be 2-isomorphic if there exists a 2-isomorphism $f:X \to Y$.
- 1.17 EXAMPLE In 2-TOP, topological spaces X and Y are 2-isomorphic iff they have the same homotopy type.
- 1.18 EXAMPLE In 2-CAT, small categories \underline{I} and \underline{J} are 2-isomorphic iff they are equivalent.

It is clear that 1.1 admits a "2-meta" formulation (cf. 0.1 and 0.9), thus 0 may be a conglomerate and C(X,Y) may be a metacategory.

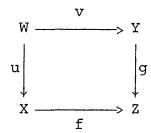
1.19 EXAMPLE There is a 2-metacategory TOP whose 0-cells are the Grothendieck toposes, whose 1-cells are the geometric morphisms, and whose 2-cells are the geometric transformations.

[Note: The 0-cells in COP constitute a conglomerate. However, if \underline{E} , \underline{F} are Grothendieck toposes and if $f,g:\underline{E}\longrightarrow \underline{F}$ are geometric morphisms, then there is just a set of natural transformations $f^*\longrightarrow g^*$ or still, just a set of geometric transformations $(f^*,f_*)\longrightarrow (g^*,g_*)$.]

1.20 NOTATION 2-CAC is the 2-metacategory whose 0-cells are the categories, whose 1-cells are the functors, and whose 2-cells are the natural transformations.

[Note: On the other hand, as agreed to above (cf. 1.12), 2-CAT is the 2-category whose 0-cells are the small categories, whose 1-cells are the functors, and whose 2-cells are the natural transformations.]

1.21 DEFINITION Let C be a 2-category -- then a diagram



of 0-cells 2-commutes (or is 2-commutative) if the 1-cells

$$\begin{array}{cccc}
 & f \circ u:W \longrightarrow Z \\
 & g \circ v:W \longrightarrow Z
\end{array}$$

are isomorphic, i.e., if there exists an invertible 2-cell ϕ in C(W,Z) such that

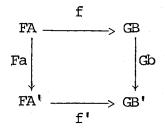
$$\phi: f \circ u \longrightarrow g \circ v.$$

1.22 EXAMPLE Given categories $\underline{A},\underline{B},\underline{C}$ and functors $F:\underline{A}\to\underline{C},\ G:\underline{B}\to\underline{C},\ \text{let }\underline{A}\succeq_{\underline{C}}\underline{B}$

 $f:FA \rightarrow GB$ is an isomorphism in C, and whose morphisms

$$(A,B,f) \longrightarrow (A',B',f')$$

are the pairs (a,b), where a:A \rightarrow A' is a morphism in \underline{A} and b:B \rightarrow B' is a morphism in B, such that the diagram



commutes. Define functors

$$\begin{bmatrix} & P: \underline{A} \times_{\underline{C}} \underline{B} \longrightarrow \underline{A} \\ & Q: \underline{A} \times_{\underline{C}} \underline{B} \longrightarrow \underline{B} \end{bmatrix}$$

by

$$P(A,B,f) = A$$
 $(P(a,b) = a)$
 $Q(A,B,f) = B$ $(P(a,b) = b)$

and define a natural isomorphism

$$E:F \circ P \longrightarrow G \circ Q$$

by

$$\Xi_{(A,B,f)}: FP(A,B,f) = FA \xrightarrow{f} GB = GQ(A,B,f).$$

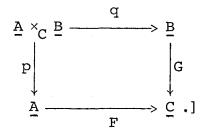
Then the diagram

of 0-cells in 2-CAT is 2-commutative.

[Note: $\underline{A} \times_{\underline{C}} \underline{B}$ is called the <u>pseudo pullback</u> of the 2-sink $\underline{A} \xrightarrow{F} \underline{C} \xleftarrow{B}$. In this connection, recall that the pullback $\underline{A} \times_{\underline{C}} \underline{B}$ of (F,G) is the category whose objects are the pairs (A,B) (A \in Ob \underline{A} , B \in Ob \underline{B}) such that FA = GB and whose morphisms

$$(A,B) \longrightarrow (A',B')$$

are the pairs (a,b), where $a:A \to A'$ is a morphism in \underline{A} and $b:\underline{B} \to \underline{B}'$ is a morphism in \underline{B} , such that Fa = Gb, there being, then, a commutative diagram



1.23 REMARK The comparison functor

$$\Gamma: \underline{A} \times_{\underline{C}} \underline{B} \longrightarrow \underline{A} \times_{\underline{C}} \underline{B}$$

is the rule that sends (A,B) to (A,B,id) (id the identity per FA = GB) and (a,b) to (a,b). While clearly fully faithful, Γ need not have a representative image, hence is not an equivalence in general.

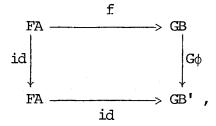
Definition: G has the <u>isomorphism lifting property</u> if \forall isomorphism $\psi:GB \rightarrow C$ in C, \exists an isomorphism $\phi:B \rightarrow B'$ in B such that $G\phi = \psi$ (so GB' = C).

Exercise: Given $G:\underline{B} \to \underline{C}$, the comparison functor Γ is an equivalence for all $F:\underline{A} \to \underline{C}$ if G has the isomorphism lifting property.

Solution: Take an object (A,B,f) in $\underline{A} \times_{\underline{C}} \underline{B}$, let $\psi: GB \to FA$ be f^{-1} , and get an isomorphism $\phi: B \to B'$ such that $G\phi = f^{-1}$ and GB' = FA — then

$$(id_{A}, \phi): (A,B,f) \longrightarrow \Gamma(A,B')$$

is an isomorphism



thus Γ has a representative image.

§2. 2-FUNCTORS

Suppose that $\mathfrak C$ and $\mathfrak C'$ are 2-categories with 0-cells 0 and 0' -- then a 2-functor $F:\mathfrak C \to \mathfrak C'$ is the specification of a rule that assigns to each 0-cell $X \in O$ a 0-cell $FX \in O'$ and the specification of a rule that assigns to each ordered pair $X,Y \in O$ a functor

$$F_{X,Y}:C(X,Y) \longrightarrow C'(FX,FY)$$

such that the diagram

commutes and the equality

$$I_{FX} = F_{X,X} \circ I_{X}$$

obtains.

[Note: The underlying functor

sends X to FX and $f:X \rightarrow Y$ to $Uf:FX \rightarrow FY$.

N.B.

(1)
$$F_{X,X}^{1} = 1_{FX}$$
;

(2)
$$F_{X,Y}id_f = id_{F_{X,Y}f};$$

(3)
$$F_{X,Z}k \circ f = F_{Y,Z}k \circ F_{X,Y}f;$$

(4)
$$F_{X,Y}\beta \bullet \alpha = F_{X,Y}\beta \bullet F_{X,Y}\alpha;$$

(5)
$$\mathbf{F}_{\mathbf{X},\mathbf{Z}}\boldsymbol{\beta} \star \boldsymbol{\alpha} = \mathbf{F}_{\mathbf{Y},\mathbf{Z}}\boldsymbol{\beta} \star \mathbf{F}_{\mathbf{X},\mathbf{Y}}\boldsymbol{\alpha}$$
.

2.1 EXAMPLE There is a 2-functor

$$II:2-TOP \longrightarrow 2-CAT$$

that sends a topological space X to its fundamental groupoid NX.

2.2 EXAMPLE Let \underline{C} and $\underline{C'}$ be locally small finitely complete categories and let $\phi:\underline{C}\to\underline{C'}$ be a functor that preserves finite limits — then there is an induced 2-functor

$$CAT(\phi):CAT(C) \longrightarrow CAT(C')$$
 (cf. 1.6).

- 2.3 NOTATION Let C be a 2-category.
- ullet $\mathfrak{c}^{1-\mathrm{OP}}$ is the 2-category obtained by reversing the 1-cells but not the 2-cells, thus

$$e^{1-OP}(x,y) = e(y,x)$$
.

 \bullet $\ensuremath{\mathfrak{C}}^{\text{2-OP}}$ is the 2-category obtained by reversing the 2-cells but not the 1-cells, thus

$$e^{2-OP}(x,y) = e(x,y)^{OP}$$
.

 \bullet $\mathfrak{C}^{1,2\text{-OP}}$ is the 2-category obtained by reversing both the 1-cells and the 2-cells, thus

$$e^{1,2-OP}(x,y) = e(y,x)^{OP}$$
.

N.B. Taking opposites defines an isomorphism

of metacategories. On the other hand, this operation does not define a 2-functor

but it does define a 2-functor

$$(2-CAT)^{2-OP} \longrightarrow 2-CAT$$

which in fact is a "2-isomorphism".

2.4 DEFINITION A derivator in the sense of Heller is a 2-functor

$$D: (2-CAT)^{1-OP} \longrightarrow 2-CAT.$$

- 2.5 EXAMPLE Fix a category \underline{C} then there is a derivator $\underline{D}_{\underline{C}}$ in the sense of Heller that sends $\underline{I} \in Ob$ \underline{CAT} to $[\underline{I},\underline{C}]$.
- 2.6 RAPPEL Let \underline{C} be a locally small category and let $W \subset Mor \ \underline{C}$ be a class of morphisms then (\underline{C}, W) is a <u>category pair</u> if W is closed under composition and contains the identities of C.
- 2.7 EXAMPLE Let (\underline{C}, W) be a category pair. Given $\underline{I} \in Ob$ \underline{CAT} , let $W_{\underline{I}} \subset Mor[\underline{I},\underline{C}]$ be the class of morphisms that are levelwise in W then

$$([\underline{\mathtt{I}},\underline{\mathtt{C}}],\omega_{\underline{\mathtt{I}}})$$

is a category pair, so it makes sense to form the localization of $[\underline{\mathbf{I}},\underline{\mathbf{C}}]$ at $\mathbf{W}_{\underline{\mathbf{I}}}$:

$$\omega_{\underline{\underline{\mathsf{I}}}}^{-1}[\underline{\underline{\mathsf{I}}},\underline{\underline{\mathsf{C}}}].$$

Define now a derivator $D_{(\underline{C},\mathcal{W})}$ in the sense of Heller by first specifying that

$$D_{(\underline{C}, w)} \underline{I} = w_{\underline{I}}^{-1} [\underline{I}, \underline{C}].$$

Next, given a functor $F: \underline{I} \rightarrow \underline{J}$, the precomposition functor

$$F^*:[J,C] \rightarrow [I,C]$$

is a morphism of category pairs (i.e., $F^*W_{\underline{J}} \subset W_{\underline{I}}$), thus there is a functor

$$\overline{\mathbf{F}^{\star}} : \mathcal{W}_{\underline{\mathbf{J}}}^{-1}[\underline{\mathbf{J}},\underline{\mathbf{C}}] \longrightarrow \mathcal{W}_{\underline{\mathbf{I}}}^{-1}[\underline{\mathbf{I}},\underline{\mathbf{C}}]\,,$$

call it $D_{(C,W)}F$, hence

$$D_{(C,W)}^{F:D}_{(C,W)} \underline{J} \longrightarrow D_{(C,W)} \underline{I}.$$

Finally, a natural transformation $\Xi:F \to G$ induces a natural transformation

$$D_{(\underline{C}, \omega)} \Xi : D_{(\underline{C}, \omega)} F \longrightarrow D_{(\underline{C}, \omega)} G.$$

2.8 REMARK A derivator in the sense of Grothendieck is a 2-functor

D:
$$(2-CAT)^{1,2-OP} \longrightarrow 2-CAC$$
.

[Note: Using opposites, one can pass back and forth between the two notions.]

- N.B. What I call a derivator (be it in the sense of Heller or Grothendieck) others call a prederivator and what I call a homotopy theory (definition omitted) others call a derivator.
- 2.9 CONSTRUCTION Suppose that ${\mathfrak C}$ is a 2-category, fix a 0-cell ${\tt X} \in {\tt O},$ and define a 2-functor

$$\Phi_X$$
: $\mathfrak{C} \longrightarrow 2$ -cat

as follows.

• Given a 0-cell $Y \in O$, let

$$\Phi_{X} Y = \mathfrak{C}(X, Y),$$

a 0-cell in 2-CAT.

• Given an ordered pair $Y,Z \in O$, let

$$(\Phi_X)_{Y,Z}$$
: $\mathcal{C}(Y,Z) \longrightarrow 2-\mathcal{C}A\mathcal{T}(\Phi_X^Y,\Phi_X^Z)$

be the functor that sends a 1-cell $g:Y \to Z$ in $\mathcal{C}(Y,Z)$ to the 1-cell

$$(\Phi_{X})_{Y,Z}g:\mathfrak{C}(X,Y) \longrightarrow \mathfrak{C}(X,Z)$$

in 2-CAT specified by the rule

$$((\Phi_{X})_{Y,Z}g)f = g \circ f$$

$$((\Phi_{X})_{Y,Z}g)\alpha = id_{g} * \alpha$$

and sends a 2-cell $\beta:g \longrightarrow g'$ in $\mathfrak{C}(Y,Z)$ to the 2-cell

$$(\Phi_{\mathbf{X}})_{\mathbf{Y},\mathbf{Z}}\beta:(\Phi_{\mathbf{X}})_{\mathbf{Y},\mathbf{Z}}\mathbf{g} \longrightarrow (\Phi_{\mathbf{X}})_{\mathbf{Y},\mathbf{Z}}\mathbf{g}'$$

specified by the rule

$$((\Phi_X)_{Y,Z}\beta)_f = \beta * id_f.$$

2.10 EXAMPLE

• Take $C = (2-CAT)^{1-OP}$ — then the construction assigns to each small category I a derivator

$$\Phi_{\mathbf{I}} : (2 - \underline{CAT})^{1-OP} \longrightarrow 2 - \mathcal{CAT}$$

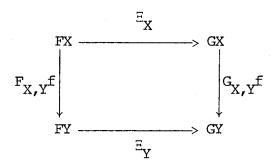
in the sense of Heller.

• Take $C = (2-CAT)^{1,2-OP}$ — then the construction assigns to each small category I a derivator

$$\Phi_{\mathbf{I}}: (2-\underline{\mathtt{CAT}})^{1,2-\mathtt{OP}} \longrightarrow 2-\mathtt{CAT}$$

in the sense of Grothendieck.

- Let \mathfrak{C} , \mathfrak{C}' be 2-categories and let $F,G:\mathfrak{C}\to\mathfrak{C}'$ be 2-functors then a 2-natural transformation $E:F\to G$ is a rule that assigns to each 0-cell $X\in O$ a 1-cell $E_X:FX\to GX$ subject to the following assumptions.
 - (1) For any 1-cell f:X → Y, the diagram



commutes.

(2) For any pair of 1-cells f,g:X \rightarrow Y and for any 2-cell α :f \Longrightarrow > g,

$$\operatorname{id}_{\Xi_{\mathbf{Y}}} * F_{\mathbf{X},\mathbf{Y}}^{\alpha} = G_{\mathbf{X},\mathbf{Y}}^{\alpha} * \operatorname{id}_{\Xi_{\mathbf{X}}}.$$

[Note: Ξ is a 2-natural isomorphism if \forall X \in O, Ξ_X is a 2-isomorphism (cf. 1.15).]

Points (1) and (2) can be rephrased.

2.11 NOTATION

• Define a functor

$$\Lambda_{F,G}: \mathfrak{C}' (FX,FY) \longrightarrow \mathfrak{C}' (FX,GY)$$

on objects by

$$\Lambda_{F,G}f' = \Xi_{Y} \circ f' \quad (f':FX \rightarrow FY)$$

and a morphism by

$$\Lambda_{F,G}^{\alpha'} = id_{\underline{E}_{Y}} * \alpha' \quad (\alpha':f' \longrightarrow g').$$

Define a functor

$$\Lambda_{G,F}: \mathfrak{C}'(GX,GY) \longrightarrow \mathfrak{C}'(FX,GY)$$

on objects by

$$\Lambda_{G,F}g' = g' \circ \Xi_{X} (g':GX \rightarrow GY)$$

and on morphisms by

$$\Lambda_{G,F}^{\beta'} = \beta' * id_{\Xi_{X}} (\beta':g' \Longrightarrow f').$$

Then it is clear that points (1) and (2) amount to the demand that the diagram

commutes.

2.12 EXAMPLE Let C and C' be locally small finitely complete categories, let $\phi, \psi: C \to C'$ be functors that preserve finite limits, and let $\xi: \phi \to \psi$ be a natural transformation — then there is an induced 2-natural transformation

$$\operatorname{CAT}(\xi): \operatorname{CAT}(\varphi) \longrightarrow \operatorname{CAT}(\psi)$$
 (cf. 2.2).

2.13 EXAMPLE Suppose that $\mathfrak C$ is a 2-category and let $f:X \to Y$ be a 1-cell -then there are 2-functors

And there is a 2-natural transformation

$$\Phi_{\mathbf{f}}:\Phi_{\mathbf{Y}}\longrightarrow\Phi_{\mathbf{X'}}$$

namely the rule that assigns to each 0-cell Z the 1-cell

$$(\Phi_{\mathsf{f}})_{\mathsf{Z}}: \mathfrak{C}(\mathsf{Y}, \mathsf{Z}) \longrightarrow \mathfrak{C}(\mathsf{X}, \mathsf{Z})$$

defined by

$$(\Phi_{\mathbf{f}})_{\mathbf{Z}} g = g \circ \mathbf{f}$$

$$(\Phi_{\mathbf{f}})_{\mathbf{Z}} \beta = \beta * \mathrm{id}_{\mathbf{f}}.$$

2.14 DEFINITION Let \mathfrak{C} , \mathfrak{C}' be 2-categories and let $F:\mathfrak{C} \to \mathfrak{C}'$ be a 2-functor -then F is a 2-equivalence if there is a 2-functor $F':\mathfrak{C}' \to \mathfrak{C}$ and 2-natural isomorphisms

$$\begin{array}{c} - & \text{F'} \circ \text{F} \longrightarrow \text{id}_{\mathfrak{C}} \\ & \text{F} \circ \text{F'} \longrightarrow \text{id}_{\mathfrak{C'}}. \end{array}$$

- 2.15 LEMMA A 2-functor $F: C \rightarrow C'$ is a 2-equivalence iff
 - (1) $\forall X,Y \in O$, the functor

$$F_{X,Y}: C(X,Y) \longrightarrow C'(FX,FY)$$

is an isomorphism of categories;

(2) $\forall X' \in O'$, $\exists X \in O$ such that FX is isomorphic to X' in UC'.

Let \mathfrak{C} , \mathfrak{C}' be 2-categories and let $F,G:\mathfrak{C}\to\mathfrak{C}'$ be 2-functors. Suppose that $E,\Omega:F\to G$ are 2-natural transformations — then a 2-modification

$$\Psi:\Xi \to \Omega$$

is a rule that assigns to each 0-cell $X \in O$ a 2-cell

$$\mathbf{q}_{\mathbf{X}} : \Xi_{\mathbf{X}} \longrightarrow \Sigma_{\mathbf{X}}$$

such that for any pair of 1-cells f,g:X \rightarrow Y and for any 2-cell α :f \Longrightarrow >g,

$$\mathbf{Y}_{\mathbf{Y}} * \mathbf{F}_{\mathbf{X},\mathbf{Y}^{\alpha}} = \mathbf{G}_{\mathbf{X},\mathbf{Y}^{\alpha}} * \mathbf{Y}_{\mathbf{X}}.$$

Let \mathfrak{C} , \mathfrak{C}' be 2-categories — then there is a 2-metacategory 2- $[\mathfrak{C},\mathfrak{C}']$ whose 0-cells are the 2-functors from \mathfrak{C} to \mathfrak{C}' , whose 1-cells are the 2-natural transformations, and whose 2-cells are the 2-modifications.

[To explicate matters:

• If $F,G:\mathbb{C} \to \mathbb{C}'$ are 2-functors, if $E,\Omega,\Gamma:F \to G$ are 2-natural transformations, and if $Y:E \to \Omega$, $M:\Omega \to \Gamma$ are 2-modifications, then $M \bullet Y:E \to \Gamma$ is defined levelwise:

$$(\mathsf{H} \bullet \mathsf{H})_{\mathsf{X}} = \mathsf{H}_{\mathsf{X}} \bullet \mathsf{H}_{\mathsf{X}}.$$

If F,G,H:¢ → ¢' are 2-functors, if

$$\begin{bmatrix} & \Xi, \Omega: F \to G \\ & \Gamma, T: G \to H \end{bmatrix}$$

are 2-natural transformations, and if $\Psi: \Xi \to \Omega$, $M: \Gamma \to T$ are 2-modifications, then $M \star \Psi: \Gamma \bullet \Xi \to T \bullet \Omega$ is defined levelwise:

$$(\mathsf{N} \star \mathsf{Y})_{\mathsf{X}} = \mathsf{N}_{\mathsf{X}} \star \mathsf{Y}_{\mathsf{X}^*}$$

2.16 EXAMPLE Let C be a 2-category -- then there is a 2-functor

$$e^{1-OP} \xrightarrow{\Phi} 2-[e,2-eat].$$

To wit:

- Send X to $\Phi_{\mathbf{X}}$ (cf. 2.9).
- Send $X \xrightarrow{f} Y$ to $\Phi_f : \Phi_Y \to \Phi_X$ (cf. 2.13).

• Send $\alpha:f \longrightarrow g$ to $\Phi_{\alpha}:\Phi_{f} \to \Phi_{g}$, where $\forall \ Z \in O$,

$$(\Phi_{\alpha})_{\mathbf{Z}}: (\Phi_{\mathbf{f}})_{\mathbf{Z}} \rightarrow (\Phi_{\mathbf{g}})_{\mathbf{Z}}$$

is the 2-natural transformation defined by stipulating that at a 1-cell $h:Y \to Z$,

$$((\Phi_{\alpha})_{z})_{h} = id_{h} * \alpha.$$

[Note:

$$\alpha: f = g$$

$$\Rightarrow id_h * \alpha: h \circ f \rightarrow h \circ g.$$

$$id_h: h = h$$

And

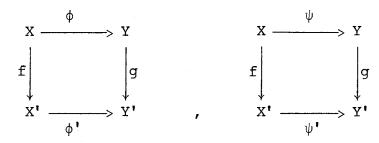
$$((\Phi_{\mathbf{f}})_{\mathbf{Z}})_{\mathbf{h}} = \mathbf{h} \circ \mathbf{f}$$

$$((\Phi_{\mathbf{g}})_{\mathbf{Z}})_{\mathbf{h}} = \mathbf{h} \circ \mathbf{g}.$$

2.17 EXAMPLE Let 2 be the category with two objects and one arrow not the identity — then if \underline{C} is a category, its arrow category $\underline{C}(\rightarrow)$ can be identified with the functor category $[\underline{2},\underline{C}]$. Now let 2 be the 2-category determined by $\underline{2}$ (cf. 1.14) — then if \underline{C} is a 2-category, we put

$$\mathcal{C}(\rightarrow) = 2-[2,\mathcal{C}].$$

Therefore the 0-cells of $\mathfrak{C}(+)$ "are" the 1-cells of \mathfrak{C} , the 1-cells of $\mathfrak{C}(+)$ "are" the commutative squares of 1-cells of \mathfrak{C} , and the 2-cells of $\mathfrak{C}(+)$ "are" the pairs



of commutative squares of 1-cells of C plus 2-cells

subject to

$$id_q * \alpha = \alpha' * id_f$$
.

[Note: The categories (UC)(\rightarrow), UC(\rightarrow) have the same objects but the first is a nonfull subcategory of the second.]

2.18 NOTATION CAT $_2$ is the 2-metacategory whose 0-cells are the 2-categories, whose 1-cells are the 2-functors, and whose 2-cells are the 2-natural transformations.

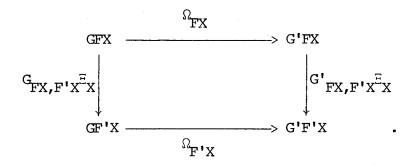
[If $E:F \to F'$ and $\Omega:G \to G'$ are 2-natural transformations, then

$$\Omega \star \Xi: G \circ F \longrightarrow G' \circ F'$$

or still,

$$(\Omega \star \Xi)_X$$
:GFX \longrightarrow G'F'X,

which in turn is defined as the corner arrow in the commutative diagram



[Note: 2-functors are composed in the obvious way.]

§3. PSEUDO FUNCTORS

Suppose that $\mathfrak C$ and $\mathfrak C'$ are 2-categories with 0-cells 0 and 0' -- then a pseudo functor $F:\mathfrak C \to \mathfrak C'$ is the specification of a rule that assigns to each 0-cell $X \in O$ a 0-cell $FX \in O'$ and the specification of a rule that assigns to each ordered pair $X,Y \in O$ a functor

$$F_{X,Y}: \mathfrak{C}(X,Y) \longrightarrow \mathfrak{C}'(FX,FY)$$

plus natural isomorphisms

$$\gamma_{X,Y,Z}:C_{FX,FY,FZ} \circ (F_{X,Y} \times F_{Y,Z}) \longrightarrow F_{X,Z} \circ C_{X,Y,Z}$$

and

$$\delta_{X}:I_{FX} \longrightarrow F_{X,X} \circ I_{X}$$

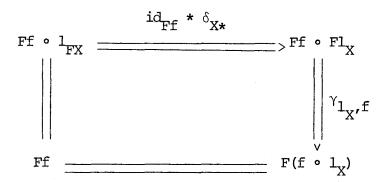
satisfying the following conditions.

 (coh_1) Given composable 1-cells f,g,h in C, the diagram

of 2-cells commutes:

$$\gamma_{g \circ f,h} \bullet (id_{Fh} * \gamma_{f,g}) = \gamma_{f,h} \circ g \bullet (\gamma_{g,h} * id_{Ff}).$$

 (coh_2) Given a 1-cell $f:X \to Y$ in \mathfrak{C} , the diagram



of 2-cells commutes:

$$\gamma_{l_{X'}f} \bullet (id_{ff} * \delta_{X*}) = id_{ff'}$$

and the diagram

of 2-cells commutes:

$$\gamma_{f,l_{Y}} \bullet (\delta_{Y*} * id_{Ff}) = id_{Ff}.$$

[Note: To ease the notational load, indices on F and γ have been suppressed, e.g., if $f:X \to Y$ and $g:Y \to Z$, then $\gamma_{f,g} = (\gamma_{X,Y,Z})_{f,g}$. Also,

stands for $\begin{bmatrix} & \delta_X \\ & & \text{evaluated at the unique object of } \underline{l}, & \text{Finally, when it is } \\ & & \delta_Y \end{bmatrix}$

necessary to exhibit the implicit dependence on F, append a superscript, e.g., $\gamma^F_{f,\sigma}, \; \delta^F_{X\star} \; \cdot]$

N.B. In $\mathfrak C$, if f,f':X —> Y, if α :f ==>f', if g,g':Y —> Z, and if β :g ==>g', then by naturality, the diagram

of 2-cells commutes:

$$F(\beta * \alpha) \bullet \gamma_{f,q} = \gamma_{f',q'} \bullet (F\beta * F\alpha).$$

- 3.1 REMARK A pseudo functor is a 2-functor iff all the $\gamma_{X,Y,Z}$ and δ_X are identities.
- 3.2 NOTATION Let MOD stand for the 2-metacategory whose 0-cells are the combinatorial model categories, whose 1-cells are the model pairs (F,F') (F a left model functor, F' a right model functor), and whose 2-cells are the natural transformations of left model functors.
 - 3.3 EXAMPLE Define a pseudo functor

$$\underline{H}:MOD \longrightarrow 2-CAT$$

as follows.

• Given a combinatorial model category C, let

$$\underline{HC} = \omega^{-1}\underline{C}$$

the localization of \underline{C} at the weak equivalences W.

ullet Given an ordered pair \underline{C} , \underline{C}' of combinatorial model categories and a model pair (F,F'), thus

send (F,F') to

$$LF: \underline{HC} \longrightarrow \underline{HC}'$$
,

where LF is the absolute total left derived functor of F.

Given a natural transformation E:F → G of left model functors, let

be the induced natural transformation of absolute total left derived functors.

3.4 NOTATION Let 2-GR stand for the 2-category whose 0-cells are the groups, whose 1-cells are the group homomorphisms, and whose 2-cells are the inner automorphisms.

[Spelled out, if G and H are groups and if f,g:G \rightarrow H are group homomorphisms, then a 2-cell α :f \Longrightarrow > g is an element α \in H such that \forall σ \in G,

$$f(\sigma)\alpha = \alpha q(\sigma)$$
.

3.5 EXAMPLE Fix a nonempty topological space B. Define a pseudo functor

$$PRIN_B: 2-\underline{GR} \longrightarrow 2-CAC$$

as follows.

ullet Given a group G, let PRIN_{B,G} be the category of principal G-spaces X over B (cf. 9.3).

Given a group homomorphism f:G → H, let

be the functor that sends X to X $\times_{\mbox{\scriptsize f}}$ H, where

$$X \times_{f} H = X \times H/\{(x \cdot \sigma, \tau) \sim (x, f(\sigma)\tau)\}.$$

• Given α :f \Longrightarrow > g, let

$$PRIN_{B,\alpha}:PRIN_{B,f} \longrightarrow PRIN_{B,g}$$

be the natural transformation which at X is the arrow

$$X \times_{f} H \longrightarrow X \times_{q} H$$

that sends (x,τ) to $(x,\alpha^{-1}\tau)$.

[Note: If $f:G \to H$, $g:H \to K$, then $\gamma_{f,g}$ is the canonical isomorphism

$$(X \times_{f} H) \times_{g} K \longrightarrow X \times_{g \circ f} K.$$

And $\delta_{\mathbf{C}_{\mathbf{x}}}$ is the canonical isomorphism

$$x \longrightarrow x \times_{id_{\mathbf{G}}} G.$$

3.6 DEFINITION Let $\mathfrak C \longrightarrow \mathfrak C' \longrightarrow \mathfrak C''$ be pseudo functors — then their composition $F' \circ F$ is the pseudo functor defined by

$$X \longrightarrow F'FX$$

and

$$(F' \circ F)_{X,Y} = F'_{FX,FY} \circ F_{X,Y}$$

plus

• Given 1-cells X \xrightarrow{f} Y and Y \xrightarrow{g} Z in \mathfrak{C} , the 2-cell γ is the f,g

composition

$$\gamma^{F'}_{Ff,Fg}$$
 $F'\gamma^{F}_{f,g}$ $F'Fg \circ F'Ff \longrightarrow F'(Fg \circ Ff) \longrightarrow F'F(g \circ f)$

and

 \bullet Given a 0-cell X in C, the 2-cell δ is the composition X*

$$1_{\text{F'FX}} \xrightarrow{\delta_{\text{FX}}^{\text{F'}}} > \text{F'}1_{\text{FX}} \xrightarrow{\text{F'}\delta_{\text{X}}^{\text{F}}} > \text{F'F1}_{\text{X}}.$$

Let \mathfrak{C} , \mathfrak{C}' be 2-categories and let $F,G:\mathfrak{C} \to \mathfrak{C}'$ be pseudo functors -- then a pseudo natural transformation $E:F \to G$ is a rule that assigns to each 0-cell $X \in O$ a 1-cell $E_X:FX \to GX$ plus a natural isomorphism

$$\tau_{X,Y}:\Lambda_{G,F} \circ G_{X,Y} \longrightarrow \Lambda_{F,G} \circ F_{X,Y}$$

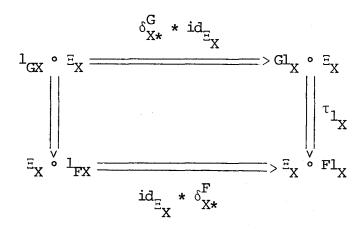
satisfying the following conditions.

(coh₁) Given 1-cells X $\stackrel{f}{-\!\!\!-\!\!\!-\!\!\!-}$ Y and Y $\stackrel{g}{-\!\!\!\!-\!\!\!\!-\!\!\!-}$ Z in $\mathfrak C$, the diagram

of 2-cells commutes:

$$(\mathrm{id}_{\Xi_{\mathbf{Z}}} * \gamma_{\mathbf{f},g}^{\mathbf{F}}) \bullet (\tau_{\mathbf{g}} * \mathrm{id}_{\mathbf{F}\mathbf{f}}) \bullet (\mathrm{id}_{\mathbf{G}\mathbf{g}} * \tau_{\mathbf{f}}) = \tau_{\mathbf{g}} \circ_{\mathbf{f}} \bullet (\gamma_{\mathbf{f},g}^{\mathbf{G}} * \mathrm{id}_{\Xi_{\mathbf{X}}}).$$

 (\cosh_2) Given a 0-cell X in \mathfrak{C} , the diagram



of 2-cells commutes:

$$\tau_{1_X} \bullet (\delta_{X^*}^G * id_{\Xi_X}) = id_{\Xi_X} * \delta_{X^*}^F.$$

(coh_3) Given 1-cells f,g:X \rightarrow Y in C and a 2-cell α :f \Longrightarrow g in C, the diagram

of 2-cells commutes:

$$(id_{\Xi_{\mathbf{Y}}} * F\alpha) \bullet \tau_{\mathbf{f}} = \tau_{\mathbf{g}} \bullet (G\alpha * id_{\Xi_{\mathbf{X}}}).$$

[Note: Again, some of the indices have been omitted.]

3.7 REMARK If $F,G:C \to C'$ are 2-functors, then a pseudo natural transformation

 Ξ is a 2-natural transformation iff all the $\tau_{X,Y}$ are identities.

3.8 DEFINITION Let $F,G,H:C \to C'$ be pseudo functors and let $E:F \to G$, $\Omega:G \to H$ be pseudo natural transformations — then their <u>composition</u> $\Omega \bullet E$ is the pseudo natural transformation defined by letting

$$(\Omega \bullet \Xi)_{\mathbf{X}} = \Omega_{\mathbf{X}} \circ \Xi_{\mathbf{X}}$$

and

$$\tau_{\mathtt{f}}^{\Omega \bullet \Xi} = (\mathrm{id}_{\Omega_{\mathtt{Y}}} \star \tau_{\mathtt{f}}^{\Xi}) \bullet (\tau_{\mathtt{f}}^{\Omega} \star \mathrm{id}_{\Xi_{\mathtt{X}}}).$$

[Note: Here τ^Ξ and τ^Ω refer to the natural transformations belonging to the pseudo natural transformations Ξ and Ω .]

3.9 REMARK There is a metacategory whose objects are the pseudo functors from $\mathfrak C$ to $\mathfrak C'$ and whose morphisms are the pseudo natural transformations.

Let \mathfrak{C} , \mathfrak{C}' be 2-categories and let $F,G:\mathfrak{C}\to\mathfrak{C}'$ be pseudo functors. Suppose that $E,\Omega:F\to G$ are pseudo natural transformations — then a pseudo modification

$$\Psi:\Xi \longrightarrow \Omega$$

is a rule that assigns to each 0-cell $X \in O$ a 2-cell

$$^{\mathsf{U}}_{\mathsf{X}}: \Xi_{\mathsf{X}} \longrightarrow ^{\mathsf{\Omega}}_{\mathsf{X}}$$

such that for any pair of 1-cells f,g:X \rightarrow Y and for any 2-cell α :f \Longrightarrow > g,

$$(\mathbf{Y}_{\mathbf{Y}} \star \mathbf{F}_{\mathbf{X},\mathbf{Y}}^{\alpha}) \bullet (\mathbf{T}_{\mathbf{X},\mathbf{Y}}^{\Xi})_{\mathbf{f}} = (\mathbf{T}_{\mathbf{X},\mathbf{Y}}^{\Omega})_{\mathbf{g}} \bullet (\mathbf{G}_{\mathbf{X},\mathbf{Y}}^{\alpha} \star \mathbf{Y}_{\mathbf{X}}).$$

3.10 REMARK If F,G: $C \to C'$ are 2-functors and if $E:F \to G$, $\Omega:F \to G$ are 2-natural

transformations, then the τ^Ξ , τ^Ω are identities and a pseudo modification $\Psi:\Xi\to\Omega$ is a 2-modification.

Pseudo modifications are composed by exactly the same procedure as 2-modifications (recall the definition of $2-[\mathfrak{C},\mathfrak{C}']$).

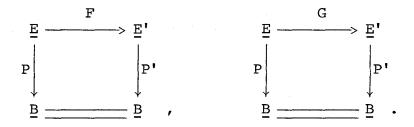
- 3.11 NOTATION PS-[$\mathfrak{C},\mathfrak{C}'$] is the 2-metacategory whose 0-cells are the pseudo functors from \mathfrak{C} to \mathfrak{C}' , whose 1-cells are the pseudo natural transformations, and whose 2-cells are the pseudo modifications.
 - N.B. 2-[C,C'] is a sub-2-metacategory of PS-[C,C'].
- 3.12 REMARK The triple consisting of 2-categories, pseudo functors, and pseudo natural transformations is not a 2-metacategory.

[Note: There is a metacategory whose objects are the 2-categories and whose morphisms are the pseudo functors.]

§4. FIBRATIONS

Fix a category \underline{B} — then the objects of CAT/\underline{B} are the pairs (\underline{E},P) , where $P:\underline{E} \to \underline{B}$ is a functor, and the morphisms $(\underline{E},P) \to (\underline{E}',P')$ of CAT/\underline{B} are the functors $F:\underline{E} \to \underline{E}'$ such that $P' \circ F = P$.

[Note: CAT/B can be regarded as a 2-metacategory, call it 2-CAT/B: Given 1-cells F,G:(E,P) \rightarrow (E',P'), a 2-cell F \Longrightarrow G is a natural transformation $\Xi:F \rightarrow G$ such that $\forall \ X \in Ob \ E$, $P'E_X = id_{PX}$. Another way to put it is this. There are commutative diagrams



And a natural transformation $\Xi:F \to G$ is a 2-cell iff

$$id_{p}$$
, * $\Xi = id_{p}$.

Here

$$id_{p}:P \rightarrow P ((id_{p})_{X} = id_{pX}).$$

Meanwhile,

$$id_{P'} * E:P' \circ F \rightarrow P' \circ G$$

and

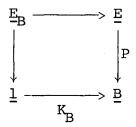
$$(id_{p'} * E)_{X} = P'E_{X}.$$

4.1 DEFINITION Let $P: \underline{E} \to \underline{B}$ be a functor and let $B \in Ob \ \underline{B}$ — then the <u>fiber</u> \underline{E}_B of P over B is the subcategory of \underline{E} whose objects are the $X \in Ob \ \underline{E}$ such that

PX = B and whose morphisms are the arrows $f \in Mor \ \underline{E}$ such that $Pf = id_{R^{\bullet}}$

[Note: In general, $\underline{\underline{E}}_B$ is not full and it may very well be the case that B and B' are isomorphic, yet $\underline{\underline{E}}_B = \underline{0}$ and $\underline{\underline{E}}_{B'} \neq \underline{0}$.]

N.B. There is a pullback square



in CAT.

- 4.2 NOTATION Given X,X' \in Ob \underline{E}_B , let $Mor_B(X,X')$ stand for the morphisms X \rightarrow X' in \underline{E}_B .
- 4.3 DEFINITION Let $X,X' \in Ob \ \underline{E}$ and let $u \in Mor(X,X')$ then u is prehorizontal if \forall morphism $w:X_0 \to X'$ of \underline{E} such that Pw = Pu, there exists a unique morphism $v \in Mor_{PX}(X_0,X)$ such that $u \circ v = w$:

$$\begin{array}{ccc}
x & \longrightarrow & X' \\
v & & \downarrow & \\
x_0 & \longrightarrow & X_0
\end{array}$$

[Note: Let

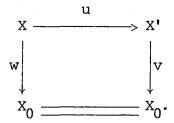
$$Mor_{ii}(X_0, X') = \{w \in Mor(X_0, X') : Pw = Pu\}.$$

Then there is an arrow

$$Mor_{px}(X_0,X) \rightarrow Mor_{n}(X_0,X^{\dagger}),$$

viz. $v \rightarrow u \circ v$ (in fact, P(u \circ v) = Pu \circ Pv = Pu \circ id_{PX} = Pu) and the condition that u be prehorizontal is that \forall X₀ \in E_{PX}, this arrow is bijective.]

4.4 DEFINITION Let $X, X' \in Ob \ \underline{E}$ and let $u \in Mor(X, X')$ — then u is preophorizontal if \forall morphism $w: X \to X_0$ of \underline{E} such that Pw = Pu, there exists a unique morphism $v \in Mor \ (X', X_0)$ such that $v \circ u = w$:



[Note: Let

$$Mor_{\mathfrak{u}}(X,X_0) = \{w \in Mor(X,X_0): Pw = Pu\}.$$

Then there is an arrow

$$\operatorname{Mor}_{\operatorname{PX}'}(\mathbf{X'},\mathbf{X}_0) \to \operatorname{Mor}_{\mathfrak{U}}(\mathbf{X},\mathbf{X}_0),$$

viz. $v \rightarrow v \circ u$ (in fact, $P(v \circ u) = Pv \circ Pu = id \circ Pu = Pu$) and the condition PX' that u be preophorizontal is that $\forall X_0 \in \underline{E}_{PX'}$ this arrow is bijective.]

- 4.5 LEMMA The isomorphisms in E are prehorizontal (preophorizontal).
- 4.6 REMARK The composite of two prehorizontal (preophorizontal) morphisms need not be prehorizontal (preophorizontal).
- 4.7 DEFINITION The functor $P: \underline{E} \to \underline{B}$ is a prefibration if for any object $X' \in Ob \ \underline{E}$ and any morphism $g: B \to PX'$, there exists a prehorizontal morphism $u: X \to X'$

such that Pu = g.

- 4.8 DEFINITION The functor $P: \underline{E} \to \underline{B}$ is a <u>preopfibration</u> if for any object $X \in Ob \ \underline{E}$ and any morphism $g: PX \to B$, there exists a preophorizontal morphism $u: X \to X'$ such that Pu = g.
- 4.9 LEMMA The functor $P: \underline{E} \to \underline{B}$ is a prefibration iff $\forall B \in Ob \ \underline{B}$, the canonical functor

$$\underline{\underline{E}}_{B} \longrightarrow B \setminus \underline{\underline{E}} \quad (X \rightarrow (id_{B}, X))$$

has a right adjoint.

4.10 LEMMA The functor $P:\underline{E}\to \underline{B}$ is a preopfibration iff \forall $B\in Ob$ \underline{B} , the canonical functor

$$\underline{\underline{E}}_{B} \longrightarrow \underline{\underline{E}}/B \quad (X \rightarrow (X, id_{B}))$$

has a left adjoint.

4.11 DEFINITION Let $X,X' \in Ob \ \underline{E}$ and let $u \in Mor(X,X')$ — then u is <u>horizontal</u> if \forall morphism $w:X_0 \to X'$ of \underline{E} and \forall factorization

$$Pw = Pu \circ x \quad (x \in Mor(PX_0, PX)),$$

there exists a unique morphism $v:X_0 \to X$ such that Pv = x and $u \circ v = w$. Schematically:

N.B. If u is horizontal, then u is prehorizontal. Proof: For Pw = Pu => $PX_0 = PX$, so we can take $x = id_{PX}$, hence $Pv = id_{PX} => v \in Mor_{PX}(X_0, X)$.

4.12 DEFINITION Let $X, X' \in Ob \ \underline{E}$ and let $u \in Mor(X, X')$ — then u is ophorizontal if \forall morphism $w: X \to X_0$ of \underline{E} and \forall factorization

$$Pw = x \circ Pu \quad (x \in Mor(PX',PX_0)),$$

there exists a unique morphism $v:X' \to X_0$ such that Pv = x and $v \circ u = w$. Schematically:

N.B. If u is ophorizontal, then u is preophorizontal. Proof: For $Pw = Pu = PX_0 = PX'$, so we can take x = id, hence Pv = id => $v \in Mor$ (X', X_0) .

4.13 DEFINITION The functor $P: \underline{E} \to \underline{B}$ is a <u>fibration</u> if for any object $X' \in Ob \ \underline{E}$ and any morphism $g: B \to PX'$, there exists a horizontal morphism $u: X \to X'$ such that Pu = g.

 $\underline{\text{N.B.}} \text{ If } \tilde{u} \colon \stackrel{\sim}{X} \to X' \text{ is another horizontal morphism such that } P\tilde{u} = \text{g, then } \exists \text{ a}$ unique isomorphism $f \in \text{Mor } \underline{E}_B$ such that $\tilde{u} = u \circ f.$

[We have

Here

Pv = id_B & u • v =
$$\tilde{u}$$
 \tilde{u}
 \tilde{v}
 \tilde{v}
 \tilde{v}
 \tilde{v}
 \tilde{v}
 \tilde{v}
 \tilde{v}
 \tilde{v}

Therefore

$$\begin{bmatrix} & \tilde{u} \circ \tilde{v} \circ v = u \circ v = \tilde{u} \\ & u \circ v \circ \tilde{v} = \tilde{u} \circ \tilde{v} = u, \end{bmatrix}$$

SO

$$\begin{array}{c|cccc}
 & \tilde{v} \circ v = id \\
 & \tilde{X} \\
 & v \circ \tilde{v} = id_{X}
\end{array}$$

4.14 DEFINITION The functor $P: \underline{E} \to \underline{B}$ is an <u>opfibration</u> if for any object $X \in Ob \ \underline{E}$ and any morphism $g: PX \to B$, there exists an ophorizontal morphism $u: X \to X'$ such that Pu = g.

 $\underline{\text{N.B.}} \text{ If } \tilde{u}\text{:}X \to \tilde{X}' \text{ is another ophorizontal morphism such that } P\tilde{u} = g \text{, then } \exists$ a unique isomorphism $f \in \text{Mor } \underline{E}_B \text{ such that } \tilde{u} = f \circ u \text{ (cf. supra).}$

4.15 LEMMA The functor $P: \underline{E} \to \underline{B}$ is a fibration iff the functor $P^{OP}: \underline{E}^{OP} \to \underline{B}^{OP}$ is an optibration.

Because of 4.15, in so far as the theory is concerned, it suffices to deal with fibrations. Still, optibrations are pervasive.

4.16 EXAMPLE The functor $\underline{E} \rightarrow \underline{l}$ is a fibration.

[Note: The functor $\underline{0} \rightarrow \underline{B}$ is a fibration (all requirements are satisfied vacuously).]

4.17 EXAMPLE The functor $\operatorname{id}_{\underline{E}}:\underline{E}\to\underline{E}$ is a fibration.

4.18 EXAMPLE Given groups
$$\begin{bmatrix} & G \\ & & \end{pmatrix}$$
 denote by $\begin{bmatrix} & G \\ & & \end{bmatrix}$ the groupoids having a \underbrace{H}

be regarded as a functor $\phi: \underline{G} \to \underline{H}$ and, as such, ϕ is a fibration iff ϕ is surjective. [Note: The fiber \underline{G}_{\star} of ϕ over \star "is" the kernel of ϕ .]

4.19 EXAMPLE Let $U:\underline{TOP} \to \underline{SET}$ be the forgetful functor — then U is a fibration. To see this, consider a morphism $g:Y \to UX'$, where Y is a set and X' is a topological space. Denote by X the topological space that arises by equipping Y with the initial topology per g (i.e., with the smallest topology such that g is continuous when viewed as a function from Y to X') — then for any topological space X_0 , a function $X_0 \to X$ is continuous iff the composition $X_0 \to X \to X'$ is continuous, from which it follows that the arrow $X \to X'$ is horizontal.

[Note: The fiber $\underline{\text{TOP}}_{Y}$ of U over Y is the partially ordered set of topologies on Y thought of as a category.]

- 4.20 LEMMA The isomorphisms in E are horizontal.
- 4.21 LEMMA Let $u \in Mor(X,X')$, $u' \in Mor(X',X'')$. Assume: u' is horizontal—then $u' \circ u$ is horizontal iff u is horizontal.

[Note: Therefore the class of horizontal morphisms is closed under composition (cf. 4.6).]

4.22 THEOREM Suppose that $P:E \to B$ is a fibration. Let $u \in Mor(X,X')$ be

horizontal. Assume: Pu is an isomorphism -- then u is an isomorphism.

PROOF In the definition of horizontal, take $\mathbf{X}_0 = \mathbf{X}'$, $\mathbf{w} = \mathrm{id}$, and consider the factorization

$$Pw = id_{PX'} = Pu \circ (Pu)^{-1} (x = (Pu)^{-1}).$$

Choose $v:X' \to X$ accordingly, thus $u \circ v = id$, so v is a right inverse for u.

But thanks to 4.20 and 4.21, v is horizontal. Since $Pv = (Pu)^{-1}$, the argument can be repeated to get a right inverse for v. Therefore u is an isomorphism.

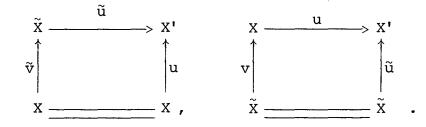
4.23 APPLICATION A fibration $P: \underline{E} \to \underline{B}$ has the isomorphism lifting property (cf. 1.23).

[Let $\psi:PX' \to B$ be an isomorphism in \underline{B} . Choose a horizontal morphism $u:X \to X'$ such that $Pu = \psi^{-1}$ — then u is an isomorphism in \underline{E} (cf. 4.22) and $Pu^{-1} = \psi$.]

4.24 LEMMA Suppose that $P: \underline{E} \to \underline{B}$ is a fibration. Consider any object $X' \in Ob \ \underline{E}$ and any morphism $g: B \to PX'$. Assume: $\tilde{u}: \tilde{X} \to X'$ is prehorizontal and $P\tilde{u} = g$ — then \tilde{u} is horizontal.

PROOF Choose a horizontal $u:X \to X'$ such that Pu = g — then u is prehorizontal so \exists a unique isomorphism $f \in Mor \xrightarrow{E}_B$ such that $\tilde{u} = u \circ f$. Therefore \tilde{u} is horizontal (cf. 4.20 and 4.21).

[Note: Here are the details. Consider the commutative diagrams

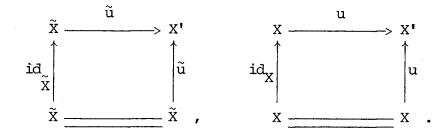


Then

$$\widetilde{\mathbf{u}} \circ \widetilde{\mathbf{v}} \circ \mathbf{v} = \mathbf{u} \circ \mathbf{v} = \widetilde{\mathbf{u}}$$

$$\mathbf{u} \circ \mathbf{v} \circ \widetilde{\mathbf{v}} = \widetilde{\mathbf{u}} \circ \widetilde{\mathbf{v}} = \mathbf{u}.$$

On the other hand, there are commutative diagrams



Therefore by the uniqueness inherent in the definition of prehorizontal,

$$\begin{array}{cccc}
 & \tilde{v} & \circ & v = id \\
 & \tilde{X} & \\
 & v & \circ & \tilde{v} = id \\
 & & & & \cdot \end{array}$$

- 4.25 THEOREM Let $P:\underline{E} \to \underline{B}$ be a functor -- then P is a fibration iff
- 1. \forall X' \in Ob \underline{E} and \forall $g \in$ Mor(B,PX'), \exists a prehorizontal $\tilde{u} \in$ Mor(\tilde{X} ,X'):P $\tilde{u} = g$ (cf. 4.7);
 - 2. The composition of two prehorizontal morphisms is prehorizontal.

PROOF The conditions are clearly necessary (for point 2, cf. 4.24 and recall 4.21). Turning to the sufficiency, one has only to prove that the \tilde{u} of point 1 is actually horizontal. Consider a morphism $w:X_0 \to X'$ of \underline{E} and a factorization

$$Pw = P\tilde{u} \circ x \quad (x \in Mor(PX_0, P\tilde{X})).$$

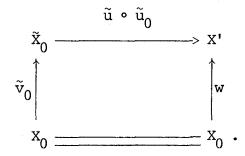
Then there is a prehorizontal $\tilde{u}_0 \in \text{Mor}(\tilde{x}_0,\tilde{x}): P\tilde{u}_0 = x \ (=> P\tilde{x}_0 = Px_0)$. Here

$$\tilde{x}_0 \xrightarrow{\tilde{u}_0} \tilde{x} \xrightarrow{\tilde{u}} x'$$

and

$$P(\tilde{u} \circ \tilde{u}_0) = P\tilde{u} \circ P\tilde{u}_0 = P\tilde{u} \circ x = Pw.$$

But $\tilde{u} \circ \tilde{u}_0$ is prehorizontal, thus there exists a unique morphism $\tilde{v}_0 \in \text{Mor}_{P\widetilde{X}_0}(x_0,\tilde{x}_0)$ such that $\tilde{u} \circ \tilde{u}_0 \circ \tilde{v}_0 = w$:



 $\text{Put } \mathbf{v} = \tilde{\mathbf{u}}_0 \circ \tilde{\mathbf{v}}_0 \text{ -- then } \mathbf{P} \mathbf{v} = \mathbf{P} \tilde{\mathbf{u}}_0 \circ \mathbf{P} \tilde{\mathbf{v}}_0 = \mathbf{P} \tilde{\mathbf{u}}_0 \circ \mathbf{id} = \mathbf{P} \tilde{\mathbf{u}}_0 = \mathbf{x} \text{ and } \tilde{\mathbf{u}} \circ \mathbf{v} = \mathbf{v}_0 = \mathbf{$

 $\tilde{u} \circ \tilde{u}_0 \circ \tilde{v}_0 = w$. To establish that v is unique, let $v': X_0 \to \tilde{X}$ be another morphism with Pv' = x and $\tilde{u} \circ v' = w$. Since \tilde{u}_0 is prehorizontal and since $Pv' = x = P\tilde{u}_0$, the diagram

admits a unique filler $v'' \in Mor_{P\widetilde{X}_0}(X_0,\widetilde{X}_0):u_0 \circ v'' = v'$. Finally

$$\tilde{\mathbf{u}} \circ \tilde{\mathbf{u}}_0 \circ \mathbf{v''} = \tilde{\mathbf{u}} \circ \mathbf{v'} = \mathbf{w}$$

$$\Rightarrow \mathbf{v''} = \tilde{\mathbf{v}}_0 \Rightarrow \mathbf{v} = \tilde{\mathbf{u}}_0 \circ \tilde{\mathbf{v}}_0 = \tilde{\mathbf{u}}_0 \circ \mathbf{v''} = \mathbf{v'}.$$

- 4.26 DEFINITION Let $P: \underline{E} \to \underline{B}$ be a functor then a morphism $f: X \to Y$ in \underline{E} is vertical if Pf is the identity on PX = PY.
 - 4.27 EXAMPLE \forall B \in Ob $\underline{\mathtt{B}}$, the elements of Mor $\underline{\mathtt{E}}_{\mathtt{B}}$ are vertical.
- 4.28 LEMMA Suppose that $P: \underline{E} \to \underline{B}$ is a fibration then every morphism in \underline{E} can be factored as a vertical morphism followed by a horizontal morphism.

PROOF Let $f:Y \to X'$ be a morphism in \underline{F} , thus $Pf:PY \to PX'$. Choose a horizontal $u:X \to X'$ such that Pu = Pf (=> PX = PY). Consider

where $Pv = id_{PX}$ (so v is vertical) and $u \circ v = f$.

- 4.29 DEFINITION A morphism $F:(\underline{E},P)\to (\underline{E}',P')$ in CAT/B is said to be <u>horizontal</u> if the functor $F:\underline{E}\to\underline{E}'$ sends horizontal arrows to horizontal arrows.
- 4.30 NOTATION CAT $_h/\underline{B}$ is the wide submetacategory of CAT/ \underline{B} whose morphisms are the horizontal morphisms.
- 4.31 NOTATION <u>FIB(B)</u> is the full submetacategory of CAC_h/B whose objects are the pairs (E,P), where $P:E \to B$ is a fibration.
 - 4.32 EXAMPLE Take $\underline{B} = \underline{1}$ -- then $\underline{FIB}(\underline{1})$ is CAT.

By definition, the 2-cells of 2-CAT/B are the <u>vertical</u> natural transformations, i.e., if F,G:(E,P) \rightarrow (E',P') are morphisms, then a 2-cell F => G is a natural

transformation $\Xi:F\to G$ such that $\forall\ X\in Ob\ \underline{E},\ P'\Xi_X=\mathrm{id}_{PX}$ or still, such that $\forall\ X\in Ob\ \underline{E},\ \Xi_X$ is a morphism in Ξ_{PX}' (P'FX = PX = P'GX), hence Ξ_X is vertical (per P').

- 4.33 NOTATION 2-CAT/B is the sub-2-metacategory of 2-CAT/B whose 0-cells are the objects of CAT/B, whose 1-cells are the horizontal morphisms, and whose 2-cells are the vertical natural transformations.
- 4.34 NOTATION FIB(B) is the 2-cell full sub-2-metacategory of 2-CAT $_h/B$ whose underlying category is FIB(B).
- 4.35 LEMMA Let $(\underline{E}_1, \underline{P}_1)$, $(\underline{E}_2, \underline{P}_2)$ be objects of CAT/B. Assume: \underline{E}_1 and \underline{E}_2 are equivalent as categories over \underline{B} , thus there are functors $F_1:\underline{E}_1 \to \underline{E}_2$ and $F_2:\underline{E}_2 \to \underline{E}_1$ over \underline{B} and vertical natural isomorphisms

$$\begin{bmatrix} & \exists_{12}: F_1 & \circ & F_2 & \longrightarrow & \mathrm{id}_{\underline{F}_2} \\ & & \vdots & & \vdots & & \vdots \\ & & & & \vdots & & \vdots \end{bmatrix}.$$

Then $\begin{bmatrix} & F_1 \\ & & \text{send horizontal arrows to horizontal arrows.} \\ & F_2 & \end{bmatrix}$

PROOF It suffices to discuss F_1 . So let $u_1: X_1 \to X_1'$ be a horizontal arrow in \underline{E}_1 , the contention being that F_1u_1 is a horizontal arrow in \underline{E}_2 . Suppose that $w_2: X_2 \to F_1X_1'$ is a morphism of \underline{E}_2 and consider a factorization

$$P_{2}^{w_{2}} = P_{2}^{F_{1}}u_{1} \circ x_{2} \quad (x_{2} \in Mor(P_{2}^{X_{2}}, P_{2}^{F_{1}}X_{1})).$$

Put

$$i = (\Xi_{21})_{X_1'}$$

thus $i:F_2F_1X_1' \longrightarrow X_1'$ and $P_1i = id_{P_1X_1'}$. Working with

$$\texttt{i} \, \circ \, \texttt{F}_2 \texttt{w}_2 \texttt{:} \texttt{F}_2 \texttt{X}_2 \, \longrightarrow \, \texttt{X}_1',$$

write

$$P_{1}(i \circ F_{2}w_{2}) = P_{1}i \circ P_{1}F_{2}w_{2}$$

$$= id_{P_{1}}x_{1}' \circ P_{2}w_{2}$$

$$= P_{2}w_{2}$$

$$= P_{2}F_{1}u_{1} \circ x_{2}$$

$$= P_{1}u_{1} \circ x_{2}.$$

Since u_1 is horizontal, there exists a unique morphism $v_1:F_2X_2 \to X_1$ such that $P_1v_1=x_2$ and $u_1 \circ v_1=i \circ F_2w_2$. Put

$$j = ((\Xi_{12})_{X_2})^{-1},$$

thus $j:X_2 \longrightarrow F_1F_2X_2$ and $P_2j = id_{P_2X_2}$. Let

$$v_2 = F_1 v_1 \circ j.$$

Then

$$P_2v_2 = P_2(F_1v_1 \circ j)$$

$$= P_2F_1v_1 \circ P_2j$$

$$= P_1v_1 \circ id_{P_2X_2}$$

$$= x_2.$$

It remains to check that

$$F_1u_1 \circ v_2 = w_2$$
.

To begin with,

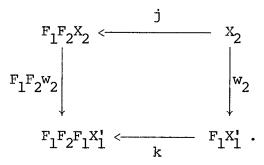
$$F_1u_1 \circ v_2 = F_1u_1 \circ F_1v_1 \circ j$$

$$= F_1(u_1 \circ v_1) \circ j$$

$$= F_1(i \circ F_2w_2) \circ j$$

$$= F_1i \circ F_1F_2w_2 \circ j.$$

On the other hand, by naturality, there is a commutative diagram



Therefore

$$F_1$$
i ° F_1 F $_2$ w $_2$ ° j = F_1 i ° k ° w $_2$ = w_2 .

Here

is the canonical arrow, hence is the identity.

[Note: The proof of uniqueness is left to the reader.]

4.36 APPLICATION $P_1:\underline{E}_1 \to \underline{B}$ is a fibration iff $P_2:\underline{E}_2 \to \underline{B}$ is a fibration.

[Suppose that P_1 is a fibration. Let $g:B \to P_2X_2'$ be a morphism in B — then the claim is that \exists a horizontal morphism $u_2:X_2 \to X_2'$ such that $P_2u_2 = g$.

- Assume first that $X_2' = F_1 X_1'$, hence $P_2 X_2' = P_2 F_1 X_1' = P_1 X_1'$, hence $g: B \to P_1 X_1'$. Choose a horizontal $u_1: X_1 \to X_1'$ such that $P_1 u_1 = g$ (=> $P_1 X_1 = B$) then $F_1 u_1: F_1 X_1 \to F_1 X_1'$ is horizontal and $P_2 F_1 u_1 = P_1 u_1 = g$, so we can take $u_2 = F_1 u_1$.
- In general, given an arbitrary X_2' , there exists an X_1' and an isomorphism $\psi: F_1X_1' \to X_2'$, from which an isomorphism $P_2\psi: P_2F_1X_1' \to P_2X_2'$ or still, an isomorphism $P_2\psi: P_1X_1' \to P_2X_2'$. If now $g: B \to P_2X_2'$, then $(P_2\psi)^{-1}: P_2X_2' \to P_1X_1'$ and, in view of what has been said above, \exists a horizontal morphism u_2 such that $P_2u_2 = (P_2\psi)^{-1} \circ g$ or still, $P_2\psi \circ P_2u_2 = g$ or still, $P_2(\psi \circ u_2) = g$. And $\psi \circ u_2$ is horizontal (cf. 4.20 and 4.21).]
- 4.37 DEFINITION Let $P: \underline{E} \to \underline{B}$, $P': \underline{E'} \to \underline{B}$ be fibrations then P, P' are equivalent if \underline{E} , $\underline{E'}$ are equivalent as categories over \underline{B} .
- N.B. If (\underline{E},P) , (\underline{E}',P') are objects of CAC/\underline{B} and if $F:(\underline{E},P) \to (\underline{E}',P')$ is a morphism, then $\forall \ B \in Ob \ \underline{B}$, F restricts to a functor $F_B:\underline{E}_B \to \underline{E}_B'$.
- 4.38 CRITERION Let $P: \underline{E} \to \underline{B}$, $P': \underline{E'} \to \underline{B}$ be fibrations, $F: (\underline{E}, P) \to (\underline{E'}, P')$ a horizontal functor then F is an equivalence of categories over \underline{B} iff \forall $B \in Ob$ \underline{B} , the functor $F_B: \underline{E}_B \to \underline{E}_B'$ is an equivalence of categories.
 - 4.39 NOTATION Given objects (\underline{E},P) , (\underline{E}',P') in $\underline{FIB}(\underline{B})$, let $[\underline{E},\underline{E}']_B$ be the

metacategory whose objects are the horizontal functors $F:(\underline{E},P)\to (\underline{E}',P')$ and whose morphisms are the vertical natural transformations.

4.40 EXAMPLE Take
$$\underline{B} = \underline{1}$$
 — then

$$[\underline{E},\underline{E}']_{\underline{1}} = [\underline{E},\underline{E}'].$$

§5. FIBRATIONS: EXAMPLES

The ensuing compilation will amply illustrate the ubiquity of the theory.

5.1 EXAMPLE The functor

that sends a small category C to its set of objects is a fibration.

[Suppose that $g:B \to Ob \ \underline{C}'$, where B is a set. To construct a horizontal $u:\underline{C} \to \underline{C}'$ such that Ob u=g, let \underline{C} have objects B and given $x,y \in B$, let

$$Mor(x,y) = \{x\} \times Mor(g(x),g(y)) \times \{y\},$$

composition and identities being those of \underline{C}' . Define the functor $u:\underline{C} \to \underline{C}'$ by taking u=g on objects and by taking

$$u:Mor(x,y) \rightarrow Mor(g(x),g(y))$$

to be the projection.]

5.2 EXAMPLE Let \underline{C} be a category with pullbacks. Consider the arrow category $\underline{C}(+)$ — then the objects of $\underline{C}(+)$ are the triples (X,f,Y), where $f:X \to Y$ is an arrow in \underline{C} , and a morphism

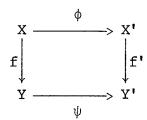
$$(X,f,Y) \rightarrow (X^{\dagger},f^{\dagger},Y^{\dagger})$$

is a pair

$$\phi: X \to X'$$

$$\psi: Y \to Y'$$

of arrows in C such that the diagram



commutes. Define

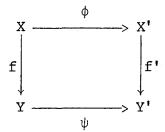
$$cod:\underline{C}(\rightarrow) \rightarrow \underline{C}$$

by

$$cod(X \longrightarrow Y) = Y, cod(\phi, \psi) = \psi.$$

Then cod is a fibration and the fiber $\underline{C}(\cdot)_Y$ of cod over Y can be identified with \underline{C}/Y .

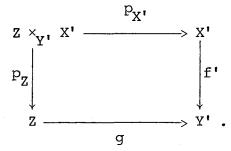
[A morphism (ϕ, ψ) is horizontal iff the commutative diagram



is a pullback square. This said, given a morphism $g:Z \to Y'$ in \underline{C} , to construct a horizontal

$$u:(X,f,Y) \rightarrow (X',f',Y')$$

such that cod u = g, form the pullback square

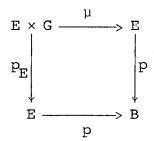


Then

$$(\textbf{p}_{X^{\textrm{!}}},\textbf{g}):(\textbf{Z}\times_{\textbf{Y}^{\textrm{!}}}\textbf{X}^{\textrm{!}},\textbf{p}_{\textbf{Z}},\textbf{Z}) \rightarrow (\textbf{X}^{\textrm{!}},\textbf{f}^{\textrm{!}},\textbf{Y}^{\textrm{!}})$$

is horizontal and $cod(p_{X'},g) = g$, so we can take $X = Z \times_{Y'} X'$, $f = p_{Z'}, Y = Z$, $u = (p_{X'},g)$.

- 5.3 EXAMPLE Let \underline{C} be a locally small finitely complete category. Fix an internal group G in \underline{C} then the restriction of cod to $G-\underline{BUN}(\underline{C})$ is a fibration. [Recall the definitions:
- An object of G-BUN(C) is an object E \xrightarrow{p} B of C/B together with an arrow E × G $\xrightarrow{\mu}$ E such that the diagram



commutes.

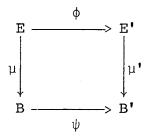
A morphism

$$(E \xrightarrow{p} B) \xrightarrow{p'} B')$$

of G-BUN(C) is a pair

$$\begin{array}{c} - & \phi: E \longrightarrow E' \\ & \psi: B \longrightarrow B' \end{array}$$

of arrows in \underline{C} such that the diagram



commutes and ϕ is G-equivariant, i.e., the diagram

commutes.]

[Note: Given a morphism $g: \tilde{B} \to B$ in C, to construct a horizontal

$$u: (\widetilde{E} \xrightarrow{\widetilde{p}} \widetilde{B}) \xrightarrow{p} B)$$

such that cod u = g, form the pullback square

$$\widetilde{E} = \widetilde{B} \times_{B} E \xrightarrow{\widetilde{g}} E$$

$$\widetilde{p} \downarrow p$$

$$\widetilde{B} \longrightarrow B$$

Then the universal property of pullback determines a unique arrow $\tilde{E}\times G \xrightarrow{\ \mu\ } \tilde{E}$ such that the diagram

$$\begin{array}{cccc}
\widetilde{E} \times G & & \widetilde{\mu} \\
p_{\widetilde{E}} & & & \widetilde{p} \\
\widetilde{E} & & & \widetilde{p}
\end{array}$$

commutes subject to

$$\tilde{g} \circ \tilde{\mu} = \mu \circ (\tilde{g} \times id_{G}).$$

Therefore $u = (\tilde{g}, g)$ is a horizontal morphism $\tilde{p} \rightarrow p$ such that cod u = g.

- 5.4 EXAMPLE Given a category \underline{C} , define a category fam \underline{C} as follows.
- \bullet The objects of fam \underline{C} are the families $\{X_{\underline{i}}: i\in I\},$ where I is a set and $X_{\underline{i}}$ \in Ob $\underline{C}.$
 - A morphism

$$\{X_{\mathbf{j}} : \mathbf{i} \in \mathbf{I}\} \rightarrow \{Y_{\mathbf{j}} : \mathbf{j} \in \mathbf{J}\}$$

of fam \underline{C} is a pair $(\phi, \{f_i : i \in I\})$, where $\phi: I \to J$ is a function and $f_i : X_i \to Y_{\phi(i)}$ is a morphism in \underline{C} .

[Note: The composite

$$(\psi,\{g_{\textbf{j}}\text{:}\text{j}\in J\}) \text{ } \circ \text{ } (\varphi,\{f_{\textbf{i}}\text{:}\text{i}\in I\})$$

is the pair

$$(\psi \circ \phi, \{g_{\phi(i)} \circ f_i : i \in I\}).]$$

Let U:fam $\underline{C} \to \underline{SET}$ be the functor that sends $\{X_i : i \in I\}$ to I and $(\phi, \{f_i : i \in I\})$ to ϕ — then U is a fibration.

[Let $\phi: I \to J$ be a function, $\{Y_j: j \in J\}$ a family of objects of \underline{C} . Put $X_i = Y_{\phi(i)}$ and let $f_i: X_i \to Y_{\phi(i)}$ be the identity — then the morphism $(\phi, \{f_i: i \in I\})$ is horizontal and its image under U is ϕ .]

[Note: The horizontal morphisms are the pairs $(\phi,\{f_{\underline{i}}\colon\! i\in I\})$, where $\forall\ i\in I$, $f_{\underline{i}}$ is an isomorphism.]

be the associated fibrations, and let $F:\underline{C}\to\underline{D}$ be a functor — then F induces a horizontal functor

fam F: fam
$$\underline{C} \rightarrow$$
 fam \underline{D}

by setting

$$\texttt{fam} \ \texttt{F} \{ \texttt{X}_{\textbf{i}} : \texttt{i} \in \texttt{I} \} = \{ \texttt{FX}_{\textbf{i}} : \texttt{i} \in \texttt{I} \}$$

and

fam
$$F(\phi, \{f_i:i \in I\}) = (\phi, \{ff_i:i \in I\})$$
.

5.5 REMARK Take $\underline{C} = \underline{SET}$ -- then the fibrations

U:fam SET
$$\rightarrow$$
 SET, cod:SET(\rightarrow) \rightarrow SET

are equivalent.

[Define a horizontal functor

on objects by sending the family $\{X_i : i \in I\}$ to the triple

$$(\coprod_{i \in I} X_i, f, I)$$
,

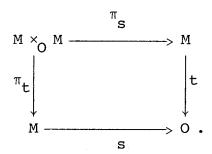
where $f(X_i) = i$, and define a horizontal functor

$$SET(\rightarrow) \rightarrow fam SET$$

on objects by sending the triple (X,f,Y) to the family $\{f^{-1}(y):y\in Y\}$.

5.6 EXAMPLE Let C be a locally small finitely complete category. Suppose that M = (M,O,s,t,e,c) is an internal category in C, thus M is an object of C, O is an object of C, and there are morphisms $s:M \to O$, $t:M \to O$, $e:O \to M$, $c:M \times_O M \to M$ satisfying the usual category theoretic relations.

Here



Define a category C(M) as follows.

- The objects of $\underline{C}(M)$ are the pairs (I,u), where I is an object of \underline{C} and $u:I \to O$ is a morphism of \underline{C} .
 - A morphism

$$(I,u) \rightarrow (J,v)$$

of C(M) is a pair (ϕ,f) , where $\phi:I \to J$ and $f:I \to M$ are morphisms of C such that $c \circ f = c \circ \phi$.

[Note: To formulate the composition law, let

$$(\phi,f):(I,u) \rightarrow (J,v), (\psi,g):(J,v) \rightarrow (K,w)$$

be morphisms. Consider the arrows

Then

$$s \circ g \circ \phi = v \circ \phi = t \circ f$$
,

from which an arrow $h:I \to M \times_O M$ such that

$$\begin{bmatrix} \pi_s \circ h = f \\ \pi_t \circ h = g \circ \phi. \end{bmatrix}$$

Now put

$$(\psi,g) \circ (\phi,f) = (\psi \circ \phi, c \circ h)$$

and observe that

$$\begin{bmatrix} - & s \circ c \circ h = s \circ \pi_s \circ h = s \circ f = u \\ \\ - & t \circ c \circ h = t \circ \pi_t \circ h = t \circ g \circ \phi = w \circ \psi \circ \phi. \end{bmatrix}$$

Let $U_M:\underline{C}(M)\to\underline{C}$ be the functor that sends (I,u) to I and (ϕ,f) to ϕ — then U_M is a fibration.

[Let $\phi: I \to J$ be a morphism of \underline{C} , where (J,v) is an object of $\underline{C}(M)$ — then the morphism

$$(\phi, e \circ v \circ \phi) : (I, v \circ \phi) \rightarrow (J, v)$$

is horizontal and its image under \textbf{U}_{M} is $\phi.]$

N.B. Let C be a locally small finitely complete category, let $\begin{bmatrix} M \\ N \end{bmatrix}$ be internal categories in C, let

$$U_{M}: \underline{C}(M) \rightarrow \underline{C}$$

$$U_{M}: \underline{C}(N) \rightarrow \underline{C}$$

be the associated fibrations, and let $F:M \to N$ be an internal functor (so $F = (F_0, F_1)$ is a pair of morphisms $F_0:O \to P$, $F_1:M \to N$ subject to ...) — then F induces a horizontal functor

$$C(F):C(M) \rightarrow C(N)$$

by setting

$$\underline{C}(F)(I,u) = (I,F_0 \circ u)$$

and

$$\underline{C}(F)(\phi,f) = (\phi,F_1 \circ f).$$

[Note: If F,G: $M \to N$ are internal functors and if $E:F \to G$ is an internal natural transformation (thought of as a morphism $E:O \to N$ subject to ...), then the prescription

$$\underline{C}(\Xi)_{(I,u)} = (id_{I}, \Xi \circ u)$$

determines a vertical natural transformation

$$C(\Xi):C(F) \rightarrow C(G)$$
.

Denote by $[M,N]_{int}$ the category whose objects are the internal functors from M to N and whose morphisms are the internal natural transformations — then the association $F \to C(F)$, $E \to C(E)$ defines a functor

$$[M,N]_{int} \rightarrow [\underline{C}(M),\underline{C}(N)]_{\underline{C}}$$
 (cf. 4.39)

which is full and faithful. Therefore, from the 2-category perspective, $CAC(\underline{C})$ (cf. 1.6) is 2-equivalent to a full sub-2-category of FIB(\underline{C}).]

5.7 REMARK Let X be an object of C. Put O = X, M = X, take $s = t = id_X$, $e = id_X$, and let X be the internal category of C thereby determined — then C(X) can be identified with C/X and U_X becomes the forgetful functor $U_X:C/X \to C$. Moreover, the functor

$$C \rightarrow FIB(C)$$

that sends X to $(\underline{C}(X),U_{\underline{X}})$ is full and faithful.

[Note: The assumption that \underline{C} is finitely complete is not needed for these considerations.]

Let \underline{I} be a small category, $F:\underline{I} \rightarrow \underline{CAT}$ a functor.

5.8 DEFINITION The integral of F over I, denoted INT_F, is the category whose objects are the pairs (i,X), where i \in Ob I and X \in Ob Fi, and whose morphisms are the arrows (δ ,f):(i,X) \rightarrow (j,Y), where δ \in Mor(i,j) and f \in Mor((F δ)X,Y) (composition is given by

$$(\delta',f') \circ (\delta,f) = (\delta' \circ \delta, f' \circ (F\delta')f)$$
.

5.9 NOTATION Let

$$\Theta_{\mathbf{F}} : \underline{\mathbf{INT}}_{\mathbf{I}} \mathbf{F} \to \underline{\mathbf{I}}$$

be the functor that sends (i,X) to i and (δ ,f) to δ .

[Note: The fiber of $\theta_{\rm F}$ over i is isomorphic to the category Fi.]

The relevant points then are these.

- The preophorizontal morphisms are the (δ,f) , where f is an isomorphism. [Note: The composition of two preophorizontal morphisms is therefore preophorizontal.]
 - ullet $\theta_{\mathbf{F}}$ is a preopfibration.

5.10 FACT $\Theta_{\rm F}$ is an opfibration (quote 4.25 in its "op" rendition).

Let $F,G:I \rightarrow CAT$ be functors, $E:F \rightarrow G$ a natural transformation.

5.11 DEFINITION The integral of Ξ over $\underline{\underline{I}}$, denoted $\underline{\underline{INT}}_{\underline{\underline{I}}}\Xi$, is the functor

$$\underline{\underline{INT}}\underline{\underline{T}}F \rightarrow \underline{\underline{INT}}\underline{\underline{T}}G$$

defined by the prescription

$$(\underline{\underline{INT}}_{\underline{\underline{I}}}\Xi)(i,X) = (i,\Xi_{\underline{i}}X)$$

$$(\underline{\underline{INT}}_{\underline{\underline{I}}}\Xi)(\delta,f) = (\delta,\Xi_{\underline{j}}f).$$

Obviously,

$$\Theta_{G} \circ \underline{INT}_{\underline{I}} \Xi = \Theta_{F}$$

and $\underline{INT}_{\underline{I}}^{\Xi}$ sends ophorizontal arrows to ophorizontal arrows. Therefore $\underline{INT}_{\underline{I}}^{\Xi}$ is an ophorizontal functor from $\underline{INT}_{\underline{I}}^{F}$ to $\underline{INT}_{\underline{I}}^{G}$.

N.B. The association

$$\begin{bmatrix} & F \rightarrow (\underline{INT}_{\underline{I}}F, \Theta_{F}) \\ & \Xi \rightarrow \underline{INT}_{\underline{I}}\Xi \end{bmatrix}$$

defines a functor

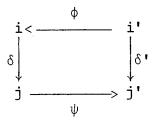
$$\underline{\text{INT}}_{\underline{\underline{\mathsf{I}}}} : [\underline{\underline{\mathsf{I}}}, \underline{\text{CAT}}] \to \underline{\text{CAT}}/\underline{\underline{\mathsf{I}}}.$$

5.12 EXAMPLE Let $\underline{\underline{I}}$ be a small category — then the <u>twisted arrow category</u> $\underline{\underline{I}}(\sim)$ of $\underline{\underline{I}}$ is the category whose objects are the triples (i,δ,j) , where $\delta:i \rightarrow j$ is an arrow in $\underline{\underline{I}}$, and a morphism

$$(i,\delta,j) \rightarrow (i',\delta',j')$$

is a pair

of arrows in I such that the diagram



commutes. Denote by $\begin{bmatrix} & s_{\underline{I}} \\ & & the \ canonical \ projections \\ & t_{\underline{I}} \end{bmatrix}$

$$\underline{\underline{I}}(\sim) \rightarrow \underline{\underline{I}}^{OP}$$

$$\underline{\underline{I}}(\sim) \rightarrow \underline{\underline{I}},$$

hence

and $\begin{bmatrix} & s_{\underline{I}} \\ & & are optibrations. \end{bmatrix}$

[Let

$$H_{\underline{I}}:\underline{\underline{I}}^{OP}\times\underline{\underline{I}}\to\underline{CAT}$$

be the functor $(j,i) \rightarrow Mor(j,i)$, where the set Mor(j,i) is regarded as a discrete category -- then

$$\frac{\text{INT}}{\text{IOP}} \times \text{I} \quad \frac{\text{H}}{\text{I}}$$

can be identified with I(~>), $\Theta_{\mbox{\scriptsize H}\underline{\mbox{\scriptsize I}}}$ becoming the functor

$$(s_{\underline{\underline{I}}}, t_{\underline{\underline{I}}}) : \underline{\underline{I}} (\sim) \rightarrow \underline{\underline{I}}^{OP} \times \underline{\underline{I}}.$$

Therefore $\begin{bmatrix} & s_{\underline{I}} \\ & & \\ & t_{\underline{I}} \end{bmatrix}$ are opfibrations (the ambient projections are opfibrations and

opfibrations are composition closed).]

The notion of pseudo pullback, as formulated in 1.22, can be extended from CAT to CAT/B.

$$F_{1}:(\underline{\mathbb{E}}_{1},P_{1}) \rightarrow (\underline{\mathbb{E}},P)$$

$$F_{2}:(\underline{\mathbb{E}}_{2},P_{2}) \rightarrow (\underline{\mathbb{E}},P)$$

be morphisms of CAT/B — then the <u>pseudo pullback</u> $E_1 \times_{\underline{E}} E_2$ of the 2-sink

$$(\underline{E}_1, P_1) \xrightarrow{F_1} (\underline{E}_2, P_2)$$

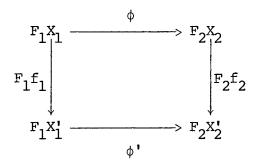
is the following category.

- An object of $\underline{\mathbf{E}}_1 \times_{\underline{\mathbf{E}}} \underline{\mathbf{E}}_2$ is a quadruple $(\mathbf{B}, \mathbf{X}_1, \mathbf{X}_2, \phi)$, where $\mathbf{B} \in \mathsf{Ob} \ \underline{\mathbf{B}}$, $\mathbf{X}_1 \in \mathsf{Ob}(\underline{\mathbf{E}}_1)_{\underline{\mathbf{B}}}$, $\mathbf{X}_2 \in \mathsf{Ob}(\underline{\mathbf{E}}_2)_{\underline{\mathbf{B}}}$, and $\phi \colon F_1 \mathbf{X}_1 \to F_2 \mathbf{X}_2$ is an isomorphism in $\underline{\mathbf{E}}_{\underline{\mathbf{B}}}$.
 - A morphism

$$(B,X_1,X_2,\phi) \longrightarrow (B',X_1',X_2',\phi')$$

is a pair (f_1, f_2) , where $f_1: X_1 \to X_1'$ is a morphism in \underline{E}_1 , $f_2: X_2 \to X_2'$ is a morphism in \underline{E}_2 , subject to f_1 and f_2 induce the same morphism $B \to B'$ (i.e., $P_1f_1 = P_2f_2$)

and the diagram



commutes.

Define functors

by

$$\begin{vmatrix} - & p_1(B, X_1, X_2, \phi) = X_1 & (p_1(f_1, f_2) = f_1) \\ & p_2(B, X_1, X_2, \phi) = X_2 & (p_2(f_1, f_2) = f_2) \end{vmatrix}$$

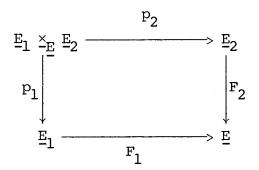
and define a natural transformation

$$\Xi:F_1 \circ P_1 \to F_2 \circ P_2$$

by

$$^{\Xi}(\mathbf{B},\mathbf{X}_{1},\mathbf{X}_{2},\phi):\mathbf{F}_{1}\mathbf{X}_{1}\xrightarrow{\phi}\mathbf{F}_{2}\mathbf{X}_{2}.$$

Then the diagram



of 0-cells in 2-CAT/B is 2-commutative.

[Note: Let

$$\Pi: \underline{E}_1 \times_{\underline{E}} \underline{E}_2 \to \underline{B}$$

be the canonical projection -- then

$$\begin{bmatrix}
F_1 & p_1: (\underline{E}_1 \times_{\underline{E}} \underline{E}_2, \Pi) \rightarrow (\underline{E}, P) \\
F_2 & p_2: (\underline{E}_1 \times_{\underline{E}} \underline{E}_2, \Pi) \rightarrow (\underline{E}, P)
\end{bmatrix}$$

are morphisms in CAT/B. E.g.:

$$\mathtt{P} \, \circ \, \mathtt{F_1} \, \circ \, \mathtt{p_1}(\mathtt{B}, \mathtt{X_1}, \mathtt{X_2}, \boldsymbol{\varphi}) \, = \, \mathtt{PF_1} \mathtt{X_1} \, = \, \mathtt{P_1} \mathtt{X_1} \, = \, \mathtt{B}$$

while

$$\Pi(B, X_1, X_2, \phi) = B.$$

Moreover, E is vertical. In fact,

$$PE_{(B,X_1,X_2,\phi)} = P\phi = id_B = id_{\Pi(B,X_1,X_2,\phi)}.$$

N.B. As regards the fibers, \forall B \in Ob B,

$$(\underline{\underline{E}}_1 \times_{\underline{\underline{E}}} \underline{\underline{E}}_2)_B \approx (\underline{\underline{E}}_1)_B \times_{\underline{\underline{E}}_B} (\underline{\underline{E}}_2)_B.$$

5.14 EXAMPLE If
$$(\underline{E}_1, \underline{P}_1)$$
, $(\underline{E}, \underline{P})$ are objects of $\underline{FIB}(\underline{B})$ and if $(\underline{E}_2, \underline{P}_2)$

$$F_{1}:(\underline{E}_{1},P_{1}) \rightarrow (\underline{E},P)$$

$$F_{2}:(\underline{E}_{2},P_{2}) \rightarrow (\underline{E},P)$$

are morphisms of $\underline{\text{FIB}}(\underline{B})$, then the canonical projection

$$\Pi \colon \underline{\mathbb{E}}_1 \, \underset{\underline{\mathbb{E}}}{\times}_{\underline{\mathbb{E}}} \, \underline{\mathbb{E}}_2 \, \to \, \underline{\mathbb{B}}$$

is a fibration.

5.15 DEFINITION The functor $P:\underline{E}\to \underline{B}$ is a <u>bifibration</u> if it is both a fibration and an opfibration.

5.16 EXAMPLE The functor

$$\texttt{Ob:}\underline{\texttt{CAT}} \to \underline{\texttt{SET}}$$

figuring in 5.1 is a bifibration.

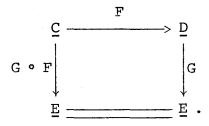
5.17 EXAMPLE The functor

$$cod:\underline{C}(\rightarrow) \rightarrow \underline{C}$$

figuring in 5.2 is a bifibration.

§6. FIBRATIONS: SORITES

- 6.1 LEMMA If $F:\underline{C} \to \underline{D}$ and $G:\underline{D} \to \underline{E}$ are fibrations, then so is their composition $G \circ F:\underline{C} \to \underline{E}$.
 - 6.2 REMARK Display the data:



Then F defines a morphism

$$(\underline{C},G \circ F) \rightarrow (\underline{D},G)$$

in CAT/E but more is true: F sends horizontal arrows to horizontal arrows. Therefore F defines a morphism

$$(\underline{C},G \circ F) \rightarrow (\underline{D},G)$$

in $\mathfrak{CAT}_{h}/\underline{\mathtt{E}}$ or still, F defines a morphism

$$(\underline{C}, G \circ F) \rightarrow (\underline{D}, G)$$

in FIB(\underline{E}).

6.3 LEMMA The projection functor

$$\overline{C} \times \overline{D} \to \overline{D}$$

is a fibration.

6.4 LEMMA If $F:\underline{C} \to \underline{D}$ and $F':\underline{C}' \to \underline{D}'$ are fibrations, then the product functor

$$F \times F^{\dagger} : \underline{C} \times \underline{C}^{\dagger} \rightarrow \underline{D} \times \underline{D}^{\dagger}$$

is a fibration.

6.5 LEMMA Let $F:\underline{C} \to \underline{D}$ be a fibration and let \underline{I} be a small category -- then

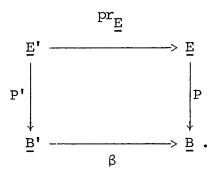
$$F_*: [\underline{I},\underline{C}] \rightarrow [\underline{I},\underline{D}]$$

is a fibration.

6.6 RAPPEL Given a 2-sink \underline{B}' \longrightarrow \underline{B} \longleftrightarrow \underline{E} in CAT, its pullback \underline{E}' = \underline{B}' $\times_{\underline{B}}$ \underline{E} is the category whose objects are the pairs (B',X) (B' \in Ob \underline{B}' , X \in Ob \underline{E}) such that $\beta B'$ = PX and whose morphisms

$$(B_1', X_1) \rightarrow (B_2', X_2)$$

are the pairs (ϕ,f) , where $\phi:B_1^{\bullet}\to B_2^{\bullet}$ is a morphism in \underline{B}^{\bullet} and $f:X_1\to X_2$ is a morphism in \underline{E} such that $\beta\phi=Pf$, there being, then, a commutative diagram



6.7 LEMMA Suppose that the functor $P: \underline{E} \to \underline{B}$ is a fibration — then for any functor $\beta: \underline{B}' \to \underline{B}$, the functor $P': \underline{E}' \to \underline{B}'$ is a fibration.

PROOF Let $g':B'' \to P'(B',X)$ (= B') be a morphism in \underline{B}' . Choose a horizontal $u:Y \to X$ such that $Pu = \beta g'$, thus $PY = \beta B''$, $PX = \beta B'$, and

$$(g',u):(B'',Y) \to (B',X)$$

is a horizontal morphism in \underline{E}' such that P'(g',u) = g'.

[Note: The opposite of a pullback square is a pullback square. So, if the functor $P: \underline{E} \to \underline{B}$ is an opfibration, then for any functor $\beta: \underline{B}' \to \underline{B}$, the functor $P': \underline{E}' \to \underline{B}'$ is an opfibration.]

N.B. The pair

$$(\underline{E}',P') \qquad \underline{FIB}(\underline{B}')$$
is an object of
$$(\underline{E},P) \qquad \underline{FIB}(\underline{B}).$$

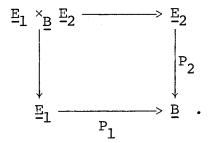
And the projection $\operatorname{pr}_{\underline{E}} \colon \underline{\underline{E}}' \to \underline{\underline{E}}$ sends horizontal arrows to horizontal arrows.

6.8 APPLICATION Suppose that

$$P_{1}:\underline{E}_{1} \to \underline{B}$$

$$P_{2}:\underline{E}_{2} \to \underline{B}$$

are fibrations. Form the pullback square



Then the corner arrow

$$\underline{E}_1 \times_{\underline{B}} \underline{E}_2 \to \underline{B}$$

is a fibration (recall 6.1).

6.9 REMARK The category $\underline{\text{FIB}}(\underline{\text{B}})$ has finite products. [The projections

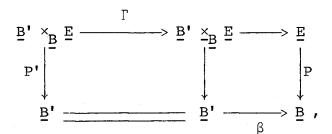
$$\begin{bmatrix} & \underline{\mathbf{E}}_1 \times_{\underline{\mathbf{B}}} \underline{\mathbf{E}}_2 \to \underline{\mathbf{E}}_1 \\ & \underline{\mathbf{E}}_1 \times_{\underline{\mathbf{B}}} \underline{\mathbf{E}}_2 \to \underline{\mathbf{E}}_2 \end{bmatrix}$$

are morphisms in $\underline{FIB}(\underline{B})$ (cf. 6.2). Therefore $\underline{FIB}(\underline{B})$ has binary products. And $\underline{id}_{\underline{B}}$ serves as a final object (cf. 4.17).]

Given a 2-sink $\underline{B}' \xrightarrow{\beta} \underline{B} < ---- \underline{E}$ in CAT, one can form its pseudo pullback $\underline{B}' \times_{\underline{B}} \underline{E}$ (cf. 1.22). Introduce the comparison functor

$$\Gamma: \underline{B}' \times_{\underline{B}} \underline{E} \to \underline{B}' \times_{\underline{B}} \underline{E}$$
 (cf. 1.23)

and consider the diagram



the square on the right being 2-commutative.

6.10 LEMMA Suppose that the functor $P: \underline{E} \to \underline{B}$ is a fibration — then the projection $\underline{B}' \times_{\underline{B}} \underline{E} \to \underline{B}'$ is a fibration.

PROOF If (B',X) is an object of $\underline{\mathtt{B}}$ ' $\times_{\underline{\mathtt{B}}}$ $\underline{\mathtt{E}}$, then

$$\Gamma(B',X) = (B',X,id) \rightarrow B' = P'(B',X)$$
.

But P has the isomorphism lifting property (cf. 4.23), hence Γ is an equivalence over \underline{B} ' (cf. 1.23), from which the assertion (cf. 4.36).

6.11 DEFINITION Let $P_1: \underline{E}_1 \to \underline{B}$, $P_2: \underline{E}_2 \to \underline{B}$ be fibrations — then a morphism $F: (\underline{E}_1, P_1) \to (\underline{E}_2, P_2) \text{ in } \underline{FIB}(\underline{B}) \text{ is said to be } \underline{internal} \text{ if given any vertical arrow}$ $f_2 \in \text{Mor } \underline{E}_2 \text{ (thus } P_2 f_2 = \text{id (cf. 4.26)), there exists a horizontal arrow } f_1 \in \text{Mor } \underline{E}_1$ per F such that $Ff_1 = f_2$ (=> $P_1 f_1 = P_2 Ff_1 = P_2 f_2 = \text{id}$).

[Note: In this context, there are three possibilities for the term "horizontal", viz. per P_1 , per P_2 , or per F_2 .

 $\underline{\text{N.B.}}$ If F is a fibration, then F is internal (recall that F is necessarily a morphism in FIB(B)).

6.12 LEMMA Suppose that F is internal -- then \forall B \in Ob B,

$$F_B: (\underline{E}_1)_B \rightarrow (\underline{E}_2)_B$$

is a fibration.

6.13 LEMMA Suppose that F is internal -- then F is a fibration.

PROOF Given a morphism $g: X_2 \to FX_1'$, the claim is that there exists a horizontal morphism $u: X_1 \to X_1'$ per F such that Fu = g. To establish this, start by applying P_2 , hence $P_2g: P_2X_2 \to P_2FX_1' = P_1X_1'$. Next, choose a horizontal morphism $\tilde{u}: \tilde{X}_1 \to X_1'$ per P_1 such that $P_1\tilde{u} = P_2g$ (=> $P_1\tilde{X}_1 = P_2X_2$) — then $F\tilde{u}$ is, by assumption, horizontal per P_2 . Consider now the factorization

$$P_{2}F\widetilde{X}_{1} \xrightarrow{id} P_{2}F\widetilde{X}_{1} \xrightarrow{P_{2}F\widetilde{u}} P_{2}FX_{1}'$$

or, equivalently, the factorization

$$\begin{array}{c|c} & & & & & \\ \hline P_2^{X_2} & & & & \\ & & id & & P_1^{\tilde{X}_1} & & & \\ & & & P_2^{F\tilde{u}} & & \end{array} \rightarrow \begin{array}{c} P_1^{X_1^{\bullet}} & \\ \end{array}$$

From the definitions, there is a unique morphism $v:X_2 \to F\widetilde{X}_1$ such that $P_2v = id$ and $F\widetilde{u} \circ v = g$. Schematically:

$$X_2 \cdot \cdot \cdot > FX_1 \xrightarrow{Fii} FX'_1$$
.

But v is vertical, so, F being internal, one can find a horizontal arrow \tilde{v} per F such that $F\tilde{v}=v$, where the codomain of \tilde{v} is \tilde{X}_1 . Put $u=\tilde{u}\circ\tilde{v}$ — then $Fu=F\tilde{u}\circ F\tilde{v}=F\tilde{u}\circ v=g$ and u is horizontal per F (verification left to the reader).]

§7. THE FUNDAMENTAL 2-EQUIVALENCE

Let \underline{B} be a category -- then \underline{B} can be regarded as a 2-category B for which $\underline{UB} \approx \underline{B}$ (cf. 1.14), but we shall abuse notation and write \underline{B} in place of B (no confusion will result in so doing).

 $\underline{N.B.}$ Traditionally, \underline{B} is replaced by \underline{B}^{OP} , the relevant 2-metacategories being

$$2-[\underline{B}^{OP}, 2-ca\tau]$$

and

$$PS-[B^{OP}, 2-CAC]$$
.

[Note: The first is a sub-2-metacategory of the second.]
The 0-cells of

$$PS-[B^{OP}, 2-CAT]$$

are the pseudo functors from \underline{B}^{OP} to 2-CAT. If $F:\underline{B}^{OP} \to 2$ -CAT is a pseudo functor, then \forall B \in Ob \underline{B} , FB is a category and \forall B, B' \in Ob \underline{B} and \forall β \in Mor(B,B'), F β :FB' \to FB is a functor.

7.1 EXAMPLE Take $\underline{B} = \underline{TOP}$ and let (X, τ_X) be a topological space — then τ_X can be viewed as a category and a continuous function $f: (X, \tau_X) \to (Y, \tau_Y)$ induces a functor $f^{-1}: \tau_Y \to \tau_X$. Therefore this data determines a 2-functor

$$\underline{\text{TOP}}^{\text{OP}} \longrightarrow 2-\mathcal{C}A\mathcal{C}.$$

7.2 EXAMPLE Take $\underline{B} = \underline{CAT}$ and fix a category \underline{D} — then for any small category \underline{C} , $[\underline{C},\underline{D}]$ is a category and a functor $F:\underline{C} \to \underline{C}'$ induces a functor $F^*:[\underline{C}',\underline{D}] \to [\underline{C},\underline{D}]$. Therefore this data determines a 2-functor

$$\underline{CAT}^{OP} \longrightarrow 2-CAC$$
.

7.3 EXAMPLE Take $\underline{B} = \underline{SCH}$ and given a scheme X, let $\underline{QCO}(X)$ be the category of quasi-coherent sheaves on X — then a morphism $f:X \to Y$ induces a functor $f^*:\underline{QCO}(Y) \to \underline{QCO}(X)$. Therefore this data determines a pseudo functor

$$\underline{\text{SCH}}^{\text{OP}} \longrightarrow 2-\text{CAT}.$$

[Note: Bear in mind that if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)*:\underline{QCO}(Z) \to \underline{QCO}(X)$ is not literally $f^* \circ g^*:\underline{QCO}(Z) \to \underline{QCO}(X)...$]

7.4 NOTATION Given pseudo functors $F,G:\underline{B}^{OP} \to 2-CAC$, let PS(F,G) stand for the metacategory whose objects are the pseudo natural transformations $\Xi:F \to G$ and whose morphisms are the pseudo modifications $\Psi:\Xi \to \Omega$.

Here is the main result.

7.5 THEOREM There is a 2-functor

$$gro_{\underline{B}}: PS-[\underline{B}^{OP}, 2-CAT] \rightarrow FIB(\underline{B})$$

with the following properties.

(1) \forall ordered pair F,G of pseudo functors $\underline{B}^{OP} \rightarrow 2-CAT$,

$$(gro_{\underline{B}})_{F,G}:PS(F,G) \rightarrow [gro_{\underline{B}}F, gro_{\underline{B}}G]_{\underline{B}}$$

is an isomorphism of metacategories.

- (2) \forall fibration $P:\underline{E} \to \underline{B}$, \exists a pseudo functor $F:\underline{B}^{OP} \to 2-\mathfrak{CAC}$ such that \underline{E} is isomorphic to $\operatorname{gro}_{\underline{B}}F$ in $\underline{FIB}(\underline{B})$.
 - 7.6 REMARK Therefore PS- $[\underline{B}^{OP}, 2-CAT]$ and $FIB(\underline{B})$ are 2-equivalent (cf. 2.15).

The proof of 7.5, when taken in all detail, is lengthy.

7.7 GROTHENDIECK CONSTRUCTION Let $F:\underline{B}^{OP} \to 2$ -CAT be a pseudo functor — then $\operatorname{gro}_{\underline{B}}F$ is the category whose objects are the pairs (B,X), where $B \in Ob \ \underline{B}$ and $X \in Ob \ FB$, and whose morphisms are the arrows $(\beta,f):(B,X) \to (B',X')$, where $\beta \in \operatorname{Mor}(B,B')$ and $f \in \operatorname{Mor}(X,(F\beta)X')$.

[Note: Suppose that

$$(\beta,f):(B,X) \to (B',X')$$
 $(\beta',f'):(B',X') \to (B'',X'').$

Then by definition

$$(\beta',f') \circ (\beta,f) = (\beta' \circ \beta, f' \circ_F f).$$

Here

$$f' \circ_{F} f \in Mor(X,F(\beta' \circ \beta)X'')$$

is the composition

f (F
$$\beta$$
)f' X \longrightarrow (F β)X' \longrightarrow (F β) (F β ')X'' \approx F(β ' \circ β)X'',

the isomorphism on the right being implicit in the definition of pseudo functor. Using the first axiom for a pseudo functor (cf. §3), one can check that this composition law is associative and using the second axiom for a pseudo functor (cf. §3), one can check that the identity in Mor((B,X),(B,X)) is the pair $(id_B,X) \approx F(id_B,X)$.

7.8 NOTATION Let

$$\Theta_{\mathbf{F}}: gro_{\underline{\mathbf{B}}}\mathbf{F} \to \underline{\mathbf{B}}$$

be the functor that sends (B,X) to B and (β ,f) to β .

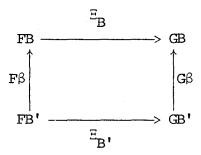
7.9 LEMMA $\boldsymbol{\theta}_F$ is a fibration and the fiber of $\boldsymbol{\theta}_F$ over B is isomorphic to the category FB.

To complete the definition of $\operatorname{gro}_{\underline{B}}$ so as to make it a 2-functor, one has to consider its action on the pseudo natural transformations and the pseudo modifications.

• Let $F,G:\underline{B}^{OP} \to 2$ -CAT be pseudo functors, $E:F \to G$ a pseudo natural transformation, the associated data thus being \forall B \in Ob B, a functor

$$\Xi_{\rm B}$$
:FB \rightarrow GB,

and $\forall \beta \in Mor(B,B')$, a 2-commutative diagram



in 2-CAT, where

$$\tau_{\beta} : \Xi_{B} \circ F\beta \longrightarrow G\beta \circ \Xi_{B}$$

is a natural isomorphism subject to the coherency conditions. We then define a horizontal functor

$$gro_{\underline{B}} \Xi : gro_{\underline{B}} F \longrightarrow gro_{\underline{B}} G$$

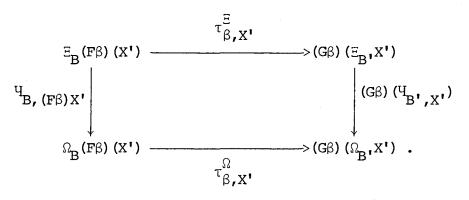
by the prescription

$$(\operatorname{gro}_{\underline{B}}^{\Xi}) (B,X) = (B,\Xi_{\underline{B}}^{X})$$
$$(\operatorname{gro}_{\underline{B}}^{\Xi}) (\beta,f) = (\beta,g),$$

where $g \in Mor(\Xi_{\mathbf{R}}X$, $(G\beta)(\Xi_{\mathbf{R}},X'))$ is the composition

$$\Xi_{\mathbf{B}}^{\mathbf{f}} \xrightarrow{\tau_{\beta,X'}} \Xi_{\mathbf{B}}^{\mathbf{f}}(\mathbf{F}\beta) (X') \xrightarrow{\tau_{\beta,X'}} (G\beta) (\Xi_{\mathbf{B}'}^{\mathbf{f}}X').$$

• Let $F,G:\underline{B}^{OP} \to 2\text{-CAT}$ be pseudo functors, $E,\Omega:F \to G$ pseudo natural transformations, and $Y:E \to \Omega$ a pseudo modification, the associated data thus being $\forall \ B \in Ob \ \underline{B}$, a natural transformation $Y_B:E_B \to \Omega_B$ subject to the commutativity of the diagram



We then define a vertical natural transformation

$$gro_{\underline{B}}^{\underline{q}}:gro_{\underline{B}}^{\underline{\Xi}} \rightarrow gro_{\underline{B}}^{\Omega}$$

by the prescription

$$(\operatorname{gro}_{B}^{\mathrm{Y}})_{(B,X)} = (\operatorname{id}_{B}^{\mathrm{Y}}_{B,X}).$$

[Note: To see that this makes sense, observe first that $gro_{\underline{B}}^{\ \ \ \ \ }$ has to be indexed by the pairs (B,X) (B \in Ob B, X \in FB), so

$$(gro_{\underline{B}}^{\underline{q}})_{(B,X)}: (gro_{\underline{B}}^{\underline{\Xi}}) (B,X) \rightarrow (gro_{\underline{B}}^{\Omega}) (B,X)$$

or still,

$$(gro_B^{\mathrm{H}})$$
 (B,X) : $(B,\Xi_B^{\mathrm{H}}X) \rightarrow (B,\Omega_B^{\mathrm{H}}X)$.

But

$$X \in FB \Rightarrow \Xi_B X \in GB$$

 $X \in FB \Rightarrow \Omega_B X \in GB.$

And $\forall X \in FB$,

$$\mathbf{Y}_{\mathbf{B},\mathbf{X}} \in \mathbf{Mor} (\mathbf{E}_{\mathbf{B}}\mathbf{X}, \Omega_{\mathbf{B}}\mathbf{X})$$
.

Therefore the pair $(id_{B}, {}^{U}_{B,X})$ belongs to

Mor(
$$(gro_B^\Xi)(B,X)$$
, $(gro_B^\Omega)(B,X)$)

per gro_B^G . That gro_B^H is vertical is obvious:

$$\begin{array}{l} \Theta_{G}(\operatorname{gro}_{\underline{B}}^{H})_{(B,X)} &= \Theta_{G}(\operatorname{id}_{B}, \ ^{H}_{B,X}) \\ \\ &= \operatorname{id}_{B} = \operatorname{id}_{\Theta_{F}(B,X)}. \end{array}$$

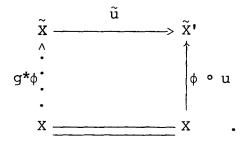
In summary: The Grothendieck construction provides us with a 2-functor

$$gro_{\underline{B}}:PS-[\underline{B}^{OP}, 2-CAT] \rightarrow FIB(\underline{B})$$

and it remains to address points (1) and (2) of 7.5. Since the verification of the first point is straightforward (albeit tedious), we shall focus on the second which requires some additional input.

Let $P: \underline{E} \to \underline{B}$ be a fibration and suppose that $g: B \to B'$ is an arrow in \underline{B} . Assuming that $\underline{E}_{B'} \neq \underline{0}$, for each $X' \in Ob \ \underline{E}_{B'}$, choose a horizontal $u: X \to X'$ such that Pu = g and define $g^*: \underline{E}_{B'} \to \underline{E}_B$ as follows.

- On an object X', let g*X' = X.
- On a morphism $\phi: X' \to \widetilde{X}'$, noting that $P(\phi \circ u) = P\phi \circ Pu = id_B$, $\circ Pu = g = P\widetilde{u}$, let $g*\phi$ be the unique filler in the fiber over B for the diagram



7.10 LEMMA $g^*:\underline{E}_{B'}\to\underline{E}_{B}$ is a functor.

[Note: Take g* to be the canonical inclusion if \underline{E}_B , = $\underline{0}$.]

Needless to say, the definition of g^* hinges on the choice of the horizontal $u:X \to X^{\bullet}$.

7.11 DEFINITION A cleavage for P is a functor σ which assigns to each pair (g,X'), where $g:B \to PX'$, a horizontal morphism $u = \sigma(g,X')$ $(u:X \to X')$ such that Pu = g.

[Note: The axiom of choice for classes implies that every fibration has a cleavage.]

7.12 EXAMPLE Consider $\text{gro}_{\underline{B}}F$ — then the <u>canonical cleavage</u> for θ_F is the rule that sends $\beta:B\to B'$ (= $\theta_F(B',X')$) to the horizontal morphism

$$(\beta, id_{(F\beta)X'}): (B, (F\beta)X') \rightarrow (B', X').$$

Consider now a pair (P,σ) , where σ is a cleavage for P -- then the association

$$B \longrightarrow \underline{\underline{E}}_{B'}$$
 $(B \longrightarrow B') \longrightarrow (\underline{\underline{E}}_{B'} \longrightarrow \underline{\underline{E}}_{B})$

defines a pseudo functor $\Sigma_{P,\sigma}$ from \underline{B}^{OP} to 2-CAT.

7.13 LEMMA If $P: \underline{E} \to \underline{B}$ is a fibration, then \underline{E} is isomorphic to $\text{gro}_{\underline{B}} \Sigma_{P,\sigma}$ in FIB(B).

PROOF Define a horizontal functor $\Phi: \underline{E} \to \operatorname{gro}_{\underline{B}} \Sigma_{P,\sigma}$ by the following procedure.

• Given $X \in Ob E$, let

$$\Phi X = (PX,X) (X \in Ob E_{PX} = Ob \Sigma_{P,O} PX)$$
.

• Given a morphism $f:Y \to X$ in \underline{F} , Φf must send $\Phi Y = (PY,Y)$ to $\Phi X = (PX,X)$. So let $\Phi f = (Pf, \varphi_f)$, where

$$\phi_{f} \in Mor(Y, (\Sigma_{P,\sigma}Pf)X),$$

or still,

$$\phi_{f} \in Mor(Y, (Pf)*X) ((Pf)*X \in E_{PY})$$

is defined to be the unique filler in the fiber over PY for the diagram

Here, by definition, $P\sigma(Pf,X) = Pf$.

The claim then is that Φ is an isomorphism of categories. But it is clear that Φ is bijective on objects. As for the morphisms, the arrow

$$Mor(Y,X) \rightarrow Mor((PY,Y),(PX,X))$$

taking f to (Pf, ϕ_f) is manifestly injective:

$$(Pf, \phi_f) = (Pg, \phi_g)$$

=>

$$f = \sigma(Pf, X) \circ \phi_f = \sigma(Pg, X) \circ \phi_g = g.$$

To establish that it is surjective, consider a pair (g,ψ) , where $g:PY \to PX$ and $\psi:Y \to (\Sigma_{P,\sigma}g)X$ (so $P\psi = id_{PY}$). Let $f = \sigma(g,PX) \circ \psi$ — then

Pf = P
$$\sigma$$
(g,PX) • P ψ
= g • id_{PY} = g.

Schematically:

$$\begin{array}{c|c} f & & Pf \\ \hline Y \cdot \cdot \cdot \cdot > g^*X \xrightarrow{\sigma(g,PX)} X & PY \xrightarrow{id_{PY}} PY \xrightarrow{g} PX \end{array} .$$

Because $\sigma(g,PX)$ is horizontal, ψ is characterized by the relations $P\psi=\mathrm{id}_{PY}$ and $\sigma(g,PX)$ \circ $\psi=f$. Meanwhile

$$Y \xrightarrow{\phi_{f}} \sigma(Pf,X) \xrightarrow{\sigma(Pf,X)} X$$

or still,

$$\begin{array}{ccc}
& & & \sigma(g,PX) \\
Y & & & & & X.
\end{array}$$

However $P\phi_f = id_{PY}$ (ϕ_f is, by definition, a morphism in the fiber over PY) and $\sigma(g,PX)$ \circ $\phi_f = f$. Accordingly, by uniqueness, $\phi_f = \psi$. Therefore

$$\Phi f = (Pf, \phi_f) = (g, \psi).$$

The proof of 7.5 is therefore complete.

§8. SPLITTINGS

Let $P: \underline{E} \rightarrow \underline{B}$ be a fibration.

- 8.1 DEFINITION A cleavage σ for P is said to be <u>split</u> if the following conditions are satisfied.
 - (1) $\sigma(\mathrm{id}_{\mathrm{p}\mathrm{X}^{1}},\mathrm{X}^{1})=\mathrm{id}_{\mathrm{X}^{1}}.$
 - (2) $\sigma(g' \circ g, X'') = \sigma(g', X'') \circ \sigma(g, g'*X'')$.

[Note: A fibration is <u>split</u> if it has a cleavage that splits or, in brief, has a splitting.]

- 8.2 EXAMPLE In the notation of 4.18, assume that $\phi: G \to H$ is surjective, hence that $\phi: G \to H$ is a fibration then a cleavage σ for ϕ is a subset K of G which maps bijectively onto H and ϕ is split iff K is a subgroup of G. Therefore ϕ is split iff ϕ is a retract, i.e., iff \exists a homomorphism $\psi: H \to G$ such that $\phi \circ \psi = \mathrm{id}_H$.
 - 8.3 REMARK The association

$$\Sigma_{P,\sigma}:\underline{B}^{OP} \rightarrow 2-CAC$$

is a 2-functor iff P is split.

8.4 THEOREM Every fibration is equivalent to a split fibration.

[Note: The meaning of the term "equivalent" is that of 4.37.]

There are some preliminaries that have to be dealt with first. So suppose that $P: \underline{E} \to \underline{B}$ is a fibration — then $\forall B \in Ob \ \underline{B}$, there is a fibration $U_{\underline{B}}: \underline{B}/B \to \underline{B}$

(cf. 5.7) and a functor

$$F_{P,B}: [\underline{B}/B,\underline{E}]_B \to \underline{E}_B,$$

namely:

(1) Given a horizontal functor

$$F: (\underline{B}/B, U_{\underline{R}}) \rightarrow (\underline{E}, P),$$

assign to F the object $F(id_B)$ in Ob \underline{E}_B .

(2) Given horizontal functors

$$F,G: (\underline{B}/B,U_{\underline{B}}) \rightarrow (\underline{E},P)$$

and a vertical natural transformation $\Xi:F \to G$, assign to Ξ the arrow $\Xi_{id_B}:F(id_B) \to G(id_B)$ in Mor \underline{E}_B .

8.5 LEMMA The functor

$$F_{P,B}: [\underline{B}/B,\underline{E}]_B \to \underline{E}_B$$

is an equivalence.

[It is not difficult to prove that $F_{P,B}$ is fully faithful. To see that $F_{P,B}$ has a representative image, fix an $X \in Ob \xrightarrow{E}_B$ and define a horizontal functor $F_X:\underline{B}/B \to \underline{F}$ by the following procedure.

• Given an object a:A \rightarrow B of B/B, put

$$F_{X}^{a} = a*X (a*: E_{B} \rightarrow E_{A} (cf. 7.10)).$$

Given a morphism

$$\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow a & & \downarrow a \\
B & \longrightarrow & B
\end{array}$$

of B/B, there are horizontal arrows

with

$$Pu = a = a' \circ f = Pu' \circ f$$

so there exists a unique morphism

$$a*f:F_Xa = a*X \longrightarrow a'*X = F_Xa'$$

such that Pa*f = f and $u' \circ a*f = u$. Schematically:

$$\begin{array}{c|c} u & & Pu \\ \hline |a*X \cdot \cdot \cdot \cdot > a'*X \xrightarrow{u'} & X \\ \hline |a*f & u' & f & A' \xrightarrow{a'} & B \\ \hline \end{array}.$$

The definitions then imply that

$$F_{P,B}F_{X} = F_{X}(id_{B})$$
$$= id_{B}^{*}X \approx X.$$

Now introduce a 2-functor

$$sp(P):\underline{B}^{OP} \rightarrow 2-CAC$$

by stipulating that

$$sp(P)(B) = [\underline{B}/B, \underline{E}]_B$$

and letting

$$sp(P)\beta:sp(P)(B') \rightarrow sp(P)(B) \quad (\beta:B \rightarrow B')$$

operate by precomposition via the horizontal arrow $\beta_{\bigstar} \colon \underline{B}/B \to \underline{B}/B'$ induced by β_{\bullet}

[Note: Strictly speaking, $[\underline{B}/B,\underline{E}]_{\underline{B}}$ is a metacategory rather than a category but this point can be safely ignored.]

Pass next to $gro_{\underline{B}}sp(P)$ — then the canonical cleavage for $\Theta_{sp(P)}$ is split (cf. 7.12).

The final step in the proof of 8.4 is to define a horizontal functor

$$F_{\underline{P}}:gro_{\underline{B}}sp(\underline{P}) \rightarrow \underline{E}$$

with the property that \forall B \in Ob \underline{B} , $(F_p)_B = F_{P,B}$. This done, it then follows from 4.38 that F_p is an equivalence of categories over B (cf. 8.5).

Consider an object (B,X) of $gro_{\underline{B}}sp(P)$ -- then

$$X \in Ob sp(P)(B) = Ob [\underline{B}/B,\underline{E}]_{B'}$$

so $X:\underline{B}/B \to \underline{E}$ is a horizontal functor and we put

$$F_{p}(B,X) = X(id_{B}) \in Ob \underline{E}_{B}.$$

Turning to a morphism $(\beta,f):(B,X)\to (B',X')$ of $gro_{\underline{B}}sp(P)$, as usual, $\beta:B\to B'$, while

$$f:X \rightarrow (sp(P)\beta)X'$$

is a vertical natural transformation indexed by the objects $A \rightarrow B$ of B/B. To define

$$F_{p}(\beta,f):X(id_{B}) \rightarrow X'(id_{B'}),$$

note first that

$$f_{id_B}:X(id_B) \rightarrow ((sp(P)\beta)X')(id_B).$$

Proceeding,

$$sp(P)\beta: [\underline{B}/B',\underline{E}]_{\underline{B}} \rightarrow [\underline{B}/B,\underline{E}]_{\underline{B}'}$$

where

$$(sp(P)\beta)X' = X' \circ \beta_*$$

hence

$$((\operatorname{sp}(P)\beta)X')(\operatorname{id}_{B}) = (X' \circ \beta_{*})(\operatorname{id}_{B})$$

$$= X'(B \xrightarrow{\beta} B').$$

In the category \underline{B}/B' , $\mathrm{id}_{\underline{B}'}:B'\to B'$ is a final object, thus there is an arrow

$$X'(B \xrightarrow{\beta} B') \longrightarrow X'(id_{B'}).$$

Definition: $F_{p}(\beta,f)$ is the result of composing

$$f_{id_B}:X(id_B) \rightarrow X'(B \xrightarrow{\beta} B')$$

with the preceding arrow, thus

$$F_p(\beta, f): X(id_B) \rightarrow X'(id_B)$$
.

§9. CATEGORIES FIBERED IN GROUPOIDS

Let $P:\underline{E} \rightarrow \underline{B}$ be a fibration.

- 9.1 DEFINITION $\underline{\mathbf{E}}$ is $\underline{\mathbf{fibered}}$ in groupoids by $\underline{\mathbf{P}}$ if \forall \mathbf{B} \in \mathbf{Ob} $\underline{\mathbf{B}}$, $\underline{\mathbf{E}}_{\underline{\mathbf{B}}}$ is a groupoid.

provided that the continuous bijection $\theta: X \times G \to X \times_{X/G} X$ defined by $(x,g) \to (x,x \cdot g)$ is a homeomorphism.

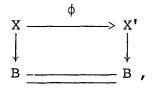
Let G be a topological group — then an X in \underline{TOP}/B is said to be a $\underline{principal}$ \underline{G} -space over B if X is a principal G-space, B is a trivial G-space, the projection $X \to B$ is open, surjective, and equivariant, and G operates transitively on the fibers. There is a commutative diagram

$$\begin{array}{ccc}
X & & & X \\
\downarrow & & \downarrow \\
X/G & & \rightarrow & B
\end{array}$$

and the arrow $X/G \rightarrow B$ is a homeomorphism.

9.3 NOTATION Let

be the category whose objects are the principal G-spaces over B and whose morphisms are the equivariant continuous functions over B, thus

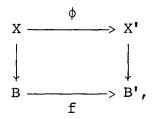


with ϕ equivariant.

9.4 FACT Every morphism in $PRIN_{B,G}$ is an isomorphism.

[Note: The objects in $\underline{PRIN}_{B,G}$ which are isomorphic to $B \times G$ (product topology) are said to be $\underline{trivial}$, thus the trivial objects are precisely those that admit a section.]

9.5 EXAMPLE Let G be a topological group — then the classifying stack of G is the category PRIN(G) whose objects are the principal G-spaces $X \to B$ and whose morphisms $(\phi, f): (X \to B) \to (X' \to B')$ are the commutative diagrams



where ϕ is equivariant. Define now a functor $P:\underline{PRIN}(G) \to \underline{TOP}$ by $P(X \to B) = B$ and $P(\phi,f) = f$ — then P is a fibration. Moreover, $\underline{PRIN}(G)$ is fibered in groupoids by P:

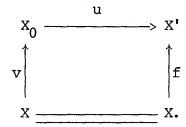
$$\underline{\underline{PRIN}}(G)_{B} = \underline{\underline{PRIN}}_{B,G'}$$

which is a groupoid by 9.4.

- 9.6 REMARK Suppose that $P: \underline{E} \to \underline{B}$ is a functor with the property that $\forall B \in Ob \ \underline{B}$, \underline{E}_B is a groupoid then it is not true in general that P is a fibration.
- [E.g.: In the notation of 4.18, consider a homomorphism $\phi:G \to H$ which is not surjective.]

9.7 LEMMA If \underline{E} is fibered in groupoids by P, then every morphism in \underline{E} is horizontal.

PROOF Let $f \in Mor(X,X')$ $(X,X' \in Ob \ \underline{E})$, thus $Pf:PX \to PX'$, so one can find a horizontal $u_0:X_0 \to X'$ such that $Pu_0 = Pf$. But u_0 is necessarily prehorizontal, hence there exists a unique morphism $v \in Mor_{PX_0}(X,X_0)$ such that $u \circ v = f$:



Since u is horizontal and v is an isomorphism, it follows that f is horizontal (cf. 4.20 and 4.11).

N.B. Suppose that

E is fibered in groupoids by P

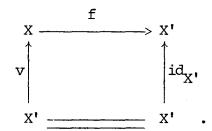
E' is fibered in groupoids by P'.

Then every functor $F: \underline{E} \to \underline{E}'$ such that $P' \circ F = P$ is automatically a horizontal functor from \underline{E} to \underline{E}' and $\left[\underline{E},\underline{E}'\right]_B$ is a groupoid.

9.8 LEMMA Let $P: \underline{E} \to \underline{B}$ be a functor. Assume: Every arrow in \underline{E} is horizontal and for any morphism $g: B \to PX'$, there exists a morphism $u: X \to X'$ such that Pu = g—then P is a fibration and \underline{E} is fibered in groupoids by P.

PROOF The conditions obviously imply that P is a fibration. Consider now an arrow $f:X \to X'$ of \underline{E}_B for some $B \in Ob$ \underline{B} — then f is horizontal, so there exists

a unique morphism $v \in Mor_B(X',X)$ (PX = B = PX') such that $f \circ v = id_{X'}$:



Therefore every arrow in \underline{E}_B has a right inverse. But this means in particular that v must have a right inverse, thus f is invertible.

9.9 LEMMA Suppose that

$$\underline{\underline{E}}_1$$
 is fibered in groupoids by \underline{P}_1 $\underline{\underline{E}}_2$ is fibered in groupoids by \underline{P}_2

and

 \underline{E} is fibered in groupoids by P.

Let

$$F_{1}:(\underline{\mathbb{E}}_{1},P_{1}) \rightarrow (\underline{\mathbb{E}},P)$$

$$F_{2}:(\underline{\mathbb{E}}_{2},P_{2}) \rightarrow (\underline{\mathbb{E}},P)$$

be morphisms in $\underline{FIB}(\underline{B})$ — then the canonical projection

$$\Pi: \underline{E}_1 \quad \underline{\times}_{\underline{E}} \ \underline{E}_2 \rightarrow \underline{B}$$

is a fibration (cf. 5.14) and $\underline{\mathbf{E}}_1 \, \, \underline{\underline{\mathbf{E}}}_2 \, \, \underline{\mathbf{E}}_2$ is fibered in groupoids by \mathbb{H} .

[Recall that

$$(\underline{\mathbb{E}}_1 \times_{\underline{\mathbb{E}}} \underline{\mathbb{E}}_2)_{\mathrm{B}} \approx (\underline{\mathbb{E}}_1)_{\mathrm{B}} \times_{\underline{\mathbb{E}}_{\mathrm{B}}} (\underline{\mathbb{E}}_2)_{\mathrm{B}}$$

and the pseudo pullback on the right is a groupoid (cf. 1.22).]

Let $P: \underline{E} \to \underline{B}$ be a fibration. Denote by \underline{E}_{hor} the wide subcategory of \underline{E} whose morphisms are the horizontal arrows of \underline{E} . Put

$$P_{hor} = P | \underline{E}_{hor}.$$

9.10 LEMMA $P_{hor}: E_{hor} \to B$ is a fibration and E_{hor} is fibered in groupoids by P_{hor} .

§10. DISCRETE FIBRATIONS

10.1 RAPPEL A category is said to be <u>discrete</u> if all its morphisms are identities.

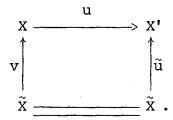
[Note: Functors between discrete categories correspond to functions on their underlying classes.]

- N.B. A discrete category is necessarily locally small.
- 10.2 EXAMPLE Every class is a discrete category and every set is a small discrete category.
- 10.3 LEMMA A category \underline{C} is equivalent to a discrete category iff \underline{C} is a groupoid with the property that \forall X,X' \in Ob \underline{C} , there is at most one morphism from X to X'.

Every discrete category is a groupoid. So, if $P: \underline{E} \to \underline{B}$ is a fibration, then the statement that \underline{E} is "fibered in discrete categories by P" (or, in brief, that \underline{E} is discretely fibered by P) is a special case of 9.1.

- 10.4 EXAMPLE Let \underline{C} be a locally small category then \forall $X \in Ob$ \underline{C} , the forgetful functor $U_{\underline{X}}:\underline{C}/X \to \underline{C}$ is a fibration (cf. 5.7). Moreover, \underline{C}/X is discretely fibered by $U_{\underline{X}}$ (\forall $Y \in Ob$ \underline{C} , the fiber (\underline{C}/X) $_{\underline{Y}}$ is the set Mor(Y,X)).
- 10.5 LEMMA Let $P: \underline{E} \to \underline{B}$ be a functor then \underline{E} is discretely fibered by P iff for any morphism $g: B \to PX'$, there exists a unique morphism $u: X \to X'$ such that Pu = g.

PROOF Assume first that \underline{F} is discretely fibered by P, choose $u:X \to X'$ per g and consider a second arrow $\tilde{u}:\tilde{X} \to X'$ per g — then $P\tilde{u}=Pu$. Since u is horizontal (cf. 9.7), thus is prehorizontal, there exists a unique morphism $v \in Mor_{PX}(\tilde{X},X)$ such that $u \circ v = \tilde{u}$:



But the fiber \underline{E}_{PX} is discrete, hence $X = \tilde{X}$ and v is the identity, so $\tilde{u} = u$. In the other direction, consider a setup

With "x" playing the role of "g", let $v:X_0 \to X$ be the unique morphism such that Pv = x — then

$$u \circ v: X_0 \to X' \implies P(u \circ v): PX_0 \to PX'$$
 $w: X_0 \to X' \implies P(w): PX_0 \to PX'.$

Accordingly, by uniqueness, $u \circ v = w$. Therefore every arrow in \underline{E} is horizontal which implies that \underline{E} is fibered in groupoids by P (cf. 9.8). That the fibers are discrete is clear.

Suppose that $P: \underline{E} \to \underline{B}$ is a fibration such that \underline{E} is fibered in sets by P (so, $\forall B \in Ob \ \underline{B}, \ \underline{E}_B$ is a set). Let $g: B \to B'$ be an arrow in \underline{B} — then the data defining the functor $g^*: \underline{E}_B$, $\to \underline{E}_B$ of 7.10 is uniquely determined, as is the cleavage

 $\sigma: P \to \Sigma_{P,\sigma}$, where in this context, $\Sigma_{P,\sigma}$ is to be viewed as a functor from \underline{E}^{OP} to SET.

10.6 NOTATION $\underline{\text{FIB}}_{\underline{\text{SET}}}(\underline{B})$ is the full subcategory of $\underline{\text{FIB}}(\underline{B})$ whose objects are the fibrations $P:\underline{E} \to \underline{B}$ which are fibered in sets by P.

If $F: (\underline{E}, P) \to (\underline{E}', P')$ is a morphism in $\underline{FIB}_{\underline{SET}}(\underline{B})$, then there is an induced natural transformation

$$\Xi_{\mathbf{F}}:\Sigma_{\mathbf{P},\sigma} \to \Sigma_{\mathbf{P}',\sigma'}$$

10.7 LEMMA The functor

$$\underline{\text{FIB}}_{\text{SET}}(\underline{\text{B}}) \rightarrow [\underline{\text{E}}^{\text{OP}},\underline{\text{SET}}]$$

that sends (\underline{E} ,P) to $\Sigma_{P,\sigma}$ is an equivalence of metacategories.

[To reverse matters, take an $F:\underline{E}^{OP}\to \underline{SET}$ and consider $gro_{\underline{B}}F$ — then here a morphism $(B,X)\to (B',X')$ is an arrow $\beta:B\to B'$ such that $X=(F\beta)X'$ and it is obvious that $gro_{\underline{B}}F$ is fibered in sets by $\Theta_{\underline{F}}$ (cf. 7.9).]

10.8 EXAMPLE Let \underline{C} be a locally small category -- then an object of

$$\hat{\underline{C}} = [\underline{C}^{OP}, \underline{SET}]$$

is called a presheaf of sets on C. Given $X \in Ob$ C, put

$$h_{X} = Mor(--,X)$$
.

Then

$$Mor(X,Y) \approx Nat(h_{X},h_{Y})$$

and in this notation the Yoneda embedding

$$Y_{\underline{C}}:\underline{C} \to \hat{\underline{C}}$$

sends X to h_{χ} . Moreover, under the correspondence of 10.7,

$$\underline{c}/x \iff h_{\underline{x}}.$$

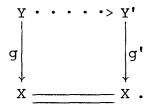
Thus, symbolically,

$$\underline{C} \longrightarrow \hat{\underline{C}} \longrightarrow \underline{\mathrm{FIB}}_{\underline{\mathrm{SET}}}(\underline{C}) \longrightarrow \underline{\mathrm{FIB}}(\underline{C}) .$$

§11. COVERING FUNCTIONS

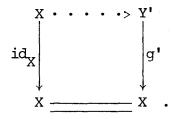
Let \underline{C} be a category.

- 11.1 DEFINITION Given an object $X \in Ob$ C, a covering of X is a subclass C of Ob C/X.
- 11.2 DEFINITION If C, C' are coverings of X, then C is a <u>refinement</u> of C' (or C refines C' or C' is refined by C) if each arrow $g \in C$ factors through an arrow $g' \in C'$:



[Note: If $C \subset C'$, then C is a refinement of C', the converse being false in general.]

11.3 EXAMPLE Take $C = \{id_X: X \to X\}$ and suppose that C is a refinement of C'—then there is an element of C' which is a split epimorphism (a.k.a. retraction):

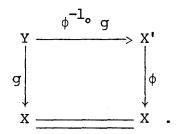


11.4 DEFINITION A covering function κ is a rule that assigns to each $X \in Ob$ C a conglomerate κ_X of coverings of X.

- 11.5 REMARK If the cardinality of Ob C/X is n, then there are 2^n subsets of Ob C/X, thus there are 2^{2^n} possible choices for κ_X .
- 11.6 NOTATION Given covering functions κ and κ' , write $\kappa' \leq \kappa$ (and term κ' subordinate to κ) if for each $X \in Ob$ \underline{C} , every covering $C' \in \kappa_X'$ is refined by some covering $C \in \kappa_X$.

11.7 EXAMPLE

- \bullet Define a covering function κ by setting κ_{X} = 0 -- then κ is subordinate to all covering functions.
- Define a covering function κ by setting κ_X = all coverings of X -- then every covering function is subordinate to κ .
- 11.8 NOTATION Given covering functions κ and κ' , write $\kappa \equiv \kappa'$ if $\kappa' \leq \kappa$ and $\kappa \leq \kappa'$, and when this is so, call κ and κ' equivalent.
- 11.9 DEFINITION Let κ be a covering function then its <u>saturation</u> is the covering function sat κ whose coverings are the coverings that have a refinement in κ .
- 11.10 EXAMPLE Assume that $\kappa_X \neq \emptyset$ and let $\phi: X' \to X$ be an isomorphism then $\{\phi\} \in (\text{sat }\kappa)_X$. Indeed, every $\mathcal{C} \in \kappa_X$ refines $\{\phi\}$:

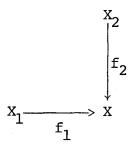


- 11.11 LEMMA Suppose that κ is a covering function then κ is equivalent to sat κ and sat κ is saturated. Moreover, κ is saturated iff κ = sat κ .
- 11.12 IEMMA Suppose that κ and κ' are covering functions then κ and κ' are equivalent iff sat κ = sat κ' .
- 11.13 DEFINITION Let κ be a covering function then κ is a <u>coverage</u> if \forall X \in Ob \underline{C} , \forall $C \in \kappa_{X'}$, and \forall $f': X' \to X$, there is a $C_{f'} \in \kappa_{X'}$, such that

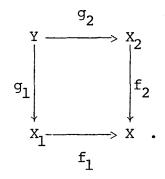
$$f' \circ C_{f'} = \{f' \circ g' : g' \in C_{f'}\} \ (Y' \xrightarrow{g'} X' \xrightarrow{f'} X)$$

is a refinement of C.

11.14 EXAMPLE Define a covering function κ by letting κ_X be comprised of all singletons $\{f\}$ ($f \in Ob \ \underline{C}/X$) — then κ is a coverage iff for each $X \in Ob \ \underline{C}$, every diagram of the form



can be completed to a commutative square

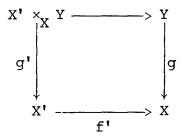


[Note: This condition is realized by the opposite of the category of finite sets and injective functions.]

11.15 LEMMA Suppose that κ and κ' are equivalent covering functions — then κ is a coverage iff κ' is a coverage.

N.B. Therefore κ is a coverage iff sat κ is a coverage (cf. 11.11).

11.16 DEFINITION Let κ be a covering function — then κ is a <u>Grothendieck</u> coverage if \forall X \in Ob C, \forall C \in $\kappa_{X'}$, \forall g:Y \rightarrow X in C, and \forall f':X' \rightarrow X, there is a pullback square



such that the covering

$$\{X' \times_X Y \xrightarrow{g'} X' : g \in C\}$$

belongs to κ_{X} .

[Note: It is a question here of a specific choice for the pullback.]

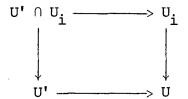
- 11.17 REMARK By construction, $f' \circ g'$ factors through g, hence a Grothendieck coverage is a coverage.
- 11.18 EXAMPLE Given a topological space X, let O(X) be the set of open subsets of X, thus under the operations

$$U \leq V \iff U \subset V, \qquad \qquad , \quad 0 = \emptyset, \quad 1 = X,$$

$$U \vee V = U \cup V$$

O(X) is a bounded lattice. Let $\underline{O}(X)$ be the category underlying O(X) and define a covering function K by stipulating that K_U is comprised of the collections $\{U_i^{-1}\}$ of open subsets U_i of U whose union U U_i is U — then K is a Grothendieck coverage. i

[Given a 2-sink U' \longrightarrow U \longleftarrow U in $\underline{O}(X)$, the commutative diagram



is a pullback square and

$$U U' \cap U_{i} = U' \cap U U_{i} = U' \cap U = U'.$$

11.19 EXAMPLE Take $\underline{C} = \underline{TOP}$ and fix $X \in Ob \ \underline{C}$. Let κ_X be comprised of the collections $\{g_i: Y_i \to X\}$ such that $\forall i$, g_i is an open map and the induced arrow $\coprod_i Y_i \to X \text{ is surjective } \text{---- then } \kappa \text{ is a Grothendieck coverage, the } \underline{open map coverage}.$

[Note: The pullback of an open map along a continuous function is an open map (in this context, "open" incorporates "continuous").]

11.20 EXAMPLE Take $\underline{C} = \underline{TOP}$ and fix $X \in Ob \underline{C}$.

• Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that \forall i, g_i is an open inclusion and the induced arrow $\coprod_i Y_i \to X$ is surjective — then κ is a Grothendieck coverage, the open subset coverage.

[Note: The pullback of an open inclusion along a continuous function is an open inclusion.]

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that \forall i, g_i is an open embedding and the induced arrow $\coprod_i Y_i \to X$ is surjective then κ is a Grothendieck coverage, the open embedding coverage.
- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that \forall i, g_i is a local homeomorphism and the induced arrow $\coprod_i Y_i \to X$ is surjective then κ is a Grothendieck coverage, the <u>local homeomorphism</u> coverage.

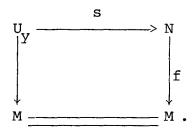
[Note: A local homeomorphism is necessarily an open map and the pullback of a local homeomorphism along a continuous function is a local homeomorphism.]

FACT The open subset coverage, the open embedding coverage, and the local homeomorphism coverage are equivalent. Moreover, each of these is subordinate to the open map coverage.

11.21 EXAMPLE Let $\underline{C^{\infty}}$ -MAN be the category whose objects are the C^{∞} -manifolds and whose morphisms are the C^{∞} -functions — then $\underline{C^{\infty}}$ -MAN does not have all pullbacks but it does have certain pullbacks, e.g., the pullback of a surjective submersion along a C^{∞} -function is again a surjective submersion. Since an open subset of a C^{∞} -manifold can be viewed as a C^{∞} -manifold, one can form the open submanifold coverage. On the other hand, there is a Grothendieck coverage K in which K_{M} is comprised of all singletons $\{f\}$, $f:N \to M$ a surjective submersion. E.g.: If $\{U_{\underline{i}}\}$ is an open submanifold coverage of M, then the induced arrow $\coprod_{\underline{i}} U_{\underline{i}} \to M$ is a surjective submersion.

[Note: If $f: N \to M$ is a surjective submersion, then $\forall \ y \in N$, there is an open subset $U_{\overset{\cdot}{Y}} \subset M$ with $f(y) \in U_{\overset{\cdot}{Y}}$ and a $C^{\overset{\circ}{\sim}}$ -function $s: U_{\overset{\cdot}{Y}} \to M$ such that $f \circ s = id$

and s(f(y)) = y:



Therefore the surjective submersion coverage is subordinate to the open submanifold coverage.]

11.22 EXAMPLE Suppose that \underline{C} has pullbacks — then there is a Grothendieck coverage κ in which $\kappa_{\underline{X}}$ is comprised of all singletons $\{f\}$ ($f \in Ob \ \underline{C}/X$), where f is a split epimorphism.

[Split epimorphisms are stable under pullback.]

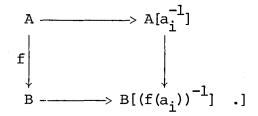
11.23 RAPPEL A locally small, finitely complete category \underline{C} fulfills the standard conditions if \underline{C} has coequalizers and the epimorphisms that are coequalizers are pullback stable.

[Note: <u>SET</u> fulfills the standard conditions (as does every topos) but <u>TOP</u> does not fulfill the standard conditions (quotient maps are not pullback stable).]

- 11.24 EXAMPLE Suppose that C fulfills the standard conditions then there is a Grothendieck coverage κ in which κ_X is comprised of all singletons $\{f\}$ ($f \in Ob \ C/X$), where f is an epimorphism that is a coequalizer.
- 11.25 DEFINITION Given an object $X \in Ob$ \underline{C} , an opcovering of X is a covering of X in \underline{C}^{OP} .

11.26 EXAMPLE Let $\underline{\mathtt{RNG}}$ be the category of commutative rings with unit. Define an opcovering function κ by letting κ_A be comprised of the collections $\{\pi_i : A \rightarrow A[a_i^{-1}]\}$, where \forall i, $A[a_i^{-1}]$ is the localization of A at a_i and the ideal generated by the set $\{a_i : i \in I\}$ is all of A — then κ is a Grothendieck opcoverage, the Zariski opcoverage.

[If $f:A \rightarrow B$ is a homomorphism, then \forall i, there is a pushout square



11.27 DEFINITION Suppose that κ is a coverage — then κ is a <u>pretopology</u> if \forall X \in Ob C, \forall C \in κ_X , \forall g:Y \rightarrow X in C, and \forall C $_g$ \in κ_Y , there is a C $_0$ \in κ_X such that C_0 is a refinement of

$$\bigcup_{g \in C} g \circ C_g = \{g \circ h : g \in C \& h \in C_g\} \ (Z \xrightarrow{h} Y \xrightarrow{g} X).$$

11.28 LEMMA If κ and κ' are equivalent coverages, then κ is a pretopology iff κ' is a pretopology.

11.29 LEMMA Suppose that κ is a pretopology. Fix X \in Ob C and let $\mathcal{C}_1,\mathcal{C}_2\in\kappa_X$ — then \exists $\mathcal{C}\in\kappa_X^{}:\mathcal{C}$ is a refinement of $\begin{matrix} & & & \\$

PROOF For each $f_2: X_2 \to X$ in C_2 , there is a $C_{f_2} \in K_{X_2}$ such that $f_2 \circ C_{f_2}$

refines \mathcal{C}_1 (cf. 11.13). On the other hand, there is a $\mathcal{C} \in \mathsf{K}_\mathsf{X}$ such that

$$\mathbf{f_2} \in \mathcal{C}_2 \quad \mathbf{f_2} \circ \ \mathcal{C}_{\mathbf{f_2}}$$

is refined by C (cf. 11.27). But

$$f_2 \in C_2$$
 $f_2 \circ C_{f_2}$

refines both C_1 and C_2 .

11.30 LEMMA Let κ be a covering function -- then κ is a pretopology iff $\kappa_{\mbox{\scriptsize sat}}$ is a pretopology.

11.31 DEFINITION Suppose that κ is a coverage — then κ is a <u>Grothendieck</u> pretopology if \forall X \in Ob \underline{C} , \forall C \in $\kappa_{\underline{X'}}$, \forall g:Y \rightarrow X in C, and \forall C_g \in $\kappa_{\underline{Y'}}$,

$$\bigcup_{g \in C} g \circ C_g = \{g \circ h : g \in C \& h \in C_g\} \ (Z \xrightarrow{h} Y \xrightarrow{g} X)$$

belongs to κ_{X} .

 ${\underline{\text{N.B.}}}$ It is obvious that a Grothendieck pretopology is a pretopology.

11.32 REMARK The various examples of Grothendieck coverages set forth above are Grothendieck pretopologies.

[The morphisms appearing in 11.22 and 11.24 are composition stable, while the verification of the requisite property in 11.26 is mildly tedious pure algebra (the terminology in this situation would be Grothendieck preoptopology...).]

[Note: Take κ per 11.14 and impose on \underline{C} the conditions therein (so that κ is a coverage) — then κ is a pretopology but it need not be a Grothendieck pretopology.]

11.33 DEFINITION A pretopology (or a Grothendieck pretopology) κ is said to have identities if \forall X \in Ob \underline{C} , $\{id_X:X \to X\}$ refines some covering in κ_X (or belongs to κ_X).

[Note: This will be the case in all examples of interest.]

11.34 REMARK If $\phi: X' \to X$ is an isomorphism in <u>C</u>, then $\{\phi\}$ might or might not belong to $\kappa_{X'}$.

[Consider the open subset coverage of 11.20 — then an arbitrary homeomorphism $\phi: X' \to X$ is certainly not admissible.]

11.35 LEMMA Let κ be a Grothendieck pretopology with the property that for any isomorphism $\phi: X' \to X$, the covering $\{\phi\}$ belongs to κ_X — then the coverings $C \in \kappa_X$ are closed under precomposition with isomorphisms, i.e., if $g: Y \to X$ is in C and if $\psi_q: Y' \to Y$ is an isomorphism, then $\{g \circ \psi_q: g \in C\} \in \kappa_X$.

PROOF By hypothesis, $\{\psi_g\} \in \kappa_{\mathrm{dom }g}$, so we can take $\mathcal{C}_g = \{\psi_g\}$, hence

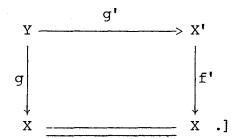
11.36 REMARK Suppose that C has pullbacks and the scenario in 11.35 is in force -- then the particular choice for the pullbacks figuring in 11.16 is immaterial.

Let κ be a covering function. Fix $X \in Ob \ \underline{C}$ — then κ induces a covering function κ on \underline{C}/X via the following procedure. Fix an object $f':X' \to X$ in \underline{C}/X — then a covering

$$\{(g:Y \longrightarrow X) \xrightarrow{g'} (F':X' \longrightarrow X)\}$$

of f' belongs to $\bar{\kappa}_{\text{f}}$ iff the covering $\{g':Y \to X'\}$ belongs to $\kappa_{X'}$.

[Note: There is a commutative diagram



N.B. If κ is a pretopology, then so is $\overline{\kappa}.$

§12. SIEVES

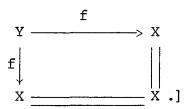
Let C be a category.

- 12.1 DEFINITION Let $X \in Ob \ \underline{C}$ then a <u>sieve</u> over X is a subclass \$ of $Ob \ \underline{C}/X$ such that the composition $Z \xrightarrow{} Y \xrightarrow{} X$ belongs to \$ if $Y \xrightarrow{} X$ belongs to \$.
 - E.g.: The <u>minimal sieve</u> over X is $\mathcal{S}_{min} = \emptyset$.
 - 12.2 LEMMA If \$ and \$' are sieves over X, then \$ refines \$' iff \$ < \$'.
- 12.3 LEMMA Every covering C of X is contained in a sieve S(C) minimal w.r.t. inclusion (the sieve generated by C).
- [$\mathfrak{F}(\mathcal{C})$ is comprised of all morphisms with codomain X which factor through some element of $\mathcal{C}.$]
 - 12.4 EXAMPLE The sieve generated by $\{id_X: X \to X\}$ is

$$s_{\text{max}} \equiv \text{Ob } \underline{C}/X$$
,

the maximal sieve over X.

[Given $f:Y \rightarrow X$, consider



It follows from 12.3 that every covering function κ gives rise to a covering function $S(\kappa)$ whose coverings at X are the $S(\mathcal{C})$ ($\mathcal{C} \in \kappa_X$).

[Note: $\$(\kappa)$ is equivalent to κ .]

12.5 DEFINITION A <u>sifted covering function</u> is a covering function all of whose coverings are sieves.

[Note: The term sifted coverage is to be assigned the obvious meaning.]

12.6 NOTATION Given a sieve \$ over X and a morphism $f:Y \to X$, put $f*\$ = \{q: cod \ q = Y \ \& \ f \circ q \in \$\}.$

Then f*\$ is a sieve over Y.

12.7 LEMMA Suppose that κ is a sifted covering function — then κ is a sifted coverage iff \forall X \in Ob \underline{C} , \forall \$ \in $\kappa_{X'}$, and \forall f':X' \rightarrow X, f'*\$ has a refinement \$'\$ in $\kappa_{X'}$.

PROOF Using the notation of 11.13, let us first prove the sufficiency of the condition. Thus put $C_{f'}=\$'$, the claim being that $f'\circ\$'$ is a refinement of \$. But

$$g' \in S' \Rightarrow g' \in f' * S (cf. 12.2) \Rightarrow f' \circ g' \in S.$$

I.e.:

$$f' \circ S' \subset S,$$

so f' \circ \$' is a refinement of \$. As for the necessity, write \$' in place of $\mathcal{C}_{f'}$, hence by assumption f' \circ \$' is a refinement of \$, hence f' \circ \$' \subset \$ (cf. 12.2) (f' \circ \$' is a sieve over X). To see that \$' \subset f'*\$, let $g' \in$ \$' — then

$$f' \circ g' \in f' \circ S' \subset S \Rightarrow g' \in f' * S.$$

12.8 DEFINITION A sifted covering function κ is sieve saturated if § $\in \kappa_X$ and § < §' => §' $\in \kappa_X$

- 12.9 LEMMA Suppose that κ is a sieve saturated sifted covering function then κ is a sifted coverage iff \forall X \in Ob \underline{C} , \forall \$ \in $\kappa_{X'}$ and \forall f':X' \rightarrow X, f'*\$ \in $\kappa_{X'}$.
- 12.10 LEMMA Suppose that κ is a sieve saturated sifted covering function then κ is a pretopology iff κ is a Grothendieck pretopology.
- 12.11 DEFINITION A sifted covering function κ is <u>locally closed</u> provided the following condition is satisfied: If $\$ \in \kappa_X$ and if \$' is a sieve over X such that $f * \$' \in \kappa_Y$ for all $f : Y \to X$ in \$, then $\$' \in \kappa_X$.
- 12.12 LEMMA Suppose that κ is a sieve saturated sifted coverage then κ is a Grothendieck pretopology iff κ is locally closed.

PROOF Using the notation of 11.31 (with "g" replaced by "f"), to check that "Grothendieck pretopology" => "locally closed",

take $\mathcal{S}_{\mathbf{f}}$ = $\mathbf{f}^*\mathcal{S}^{\mathsf{I}}$ $\in \kappa_{\mathbf{Y}}$ — then

belongs to κ_{χ} . But

$$h \in f*S' \Rightarrow f \circ h \in S'$$

$$=> \bigcup_{f \in S} f \circ S_f \subset S'.$$

Therefore $S' \in K_X$ (K being sieve saturated), so K is locally closed. Turning to the converse, the data is the sieve

$$\mathfrak{S}' = \{f \circ h: f \in \mathfrak{S} \& h \in \mathfrak{S}_f\}$$

and the claim is that it belongs to $\kappa_{\chi^{\bullet}}$. But \forall f \in \$,

$$f*\mathfrak{F}' \supset \mathfrak{F}_{\mathbf{f}} \in \kappa_{\mathbf{Y}} \Longrightarrow f*\mathfrak{F}' \in \kappa_{\mathbf{Y}}$$

$$\Longrightarrow \mathfrak{F}' \in \kappa_{\mathbf{Y}}.$$

12.13 LEMMA Let κ be a sifted covering function. Assume: κ is locally closed and \forall X \in Ob \underline{C} , $\$_{max}$ \in κ_{X} — then κ is sieve saturated.

PROOF Fix $\$ \in \mathsf{K}_{X'}$ suppose that $\$ \subset \$'$, and let $f \colon Y \to X$ be an element of \$ —then

But

$$f * \$ = Ob \underline{C} / Y \in \kappa_Y \Rightarrow f * \$' \in \kappa_Y$$
$$=> \$' \in \kappa_Y.$$

12.14 DEFINITION Suppose that κ is a sifted coverage — then κ is a <u>Grothendieck topology</u> if it is locally closed and \forall X \in Ob \underline{C} , $\$_{max} \in \kappa_{X}$.

[Note: It follows from 12.13 that κ is sieve saturated. Therefore κ is a Grothendieck pretopology (cf. 12.12) and it is automatic that 12.9 is in force.]

12.15 LEMMA If κ is a Grothendieck topology and if $\mathfrak{F},\mathfrak{F}'\in\kappa_{X'}$, then $\mathfrak{F}\cap\mathfrak{F}'\in\kappa_{X'}$. PROOF For any $f:Y\to X$ in $\mathfrak{F},$

$$f*S' = f*(S \cap S')$$
.

However, thanks to 12.9 (applied to §'), f*§' $\in \kappa_{\underline{Y}}$, so

$$\texttt{f*($^{\varsigma} \cap $^{\varsigma}'$)} \in \kappa_{\underline{Y}} \Rightarrow \$ \cap \$' \in \kappa_{\underline{X}^{\bullet}}$$

12.16 EXAMPLE Take $\underline{C} = \underline{O}(X)$, X a topological space (cf. 11.18). Given an open

set $U \subset X$, a sieve S over U is a set of open subsets V of U which is hereditary in the sense that

$$V \in \mathcal{S} \& V' \subset V \Rightarrow V' \in \mathcal{S}.$$

One then says that § covers U if U V = U. Denoting by κ_U the set of all such §, V=§ the assignment U $\rightarrow \kappa_U$ is a Grothendieck topology κ on Q(X).

- 12.17 DEFINITION Let κ be a sifted covering function then its <u>sifted</u> saturation is the sifted covering function sif κ whose coverings are the sieves that contain a sieve in κ .
 - 12.18 LEMMA For any covering function κ ,

sif
$$\S(\kappa) = \S(\text{sat }\kappa)$$
.

Denote this covering function by $J(\kappa)$ — then $J(\kappa)$ is sifted and sieve saturated.

- 12.19 LEMMA Suppose that § is a sieve over X -- then § \in J(κ) $_X$ iff § contains an element of κ_X .
- 12.20 THEOREM If κ is a pretopology with identities (cf. 11.33), then $J(\kappa)$ is a Grothendieck topology.

PROOF The assumption that κ is a pretopology implies that sat κ is a pretopology (cf. 11.28) (κ and sat κ are equivalent), hence that $\mathfrak{F}(\operatorname{sat} \kappa)$ is a pretopology (cf. 11.28) (sat κ and $\mathfrak{F}(\operatorname{sat} \kappa)$ are equivalent), in particular $J(\kappa) = \mathfrak{F}(\operatorname{sat} \kappa)$ is a coverage. Therefore $J(\kappa)$ is a Grothendieck pretopology (cf. 12.10) ($J(\kappa)$ is sieve saturated), thus $J(\kappa)$ is locally closed (cf. 12.12). Finally, if $\{\operatorname{id}_X: X \to X\}$ refines $C \in \kappa_X$, then

$$\mathfrak{F}(\{\mathrm{id}_{\mathtt{X}}:\mathtt{X}\to\mathtt{X}\})\subset\mathfrak{F}(\mathcal{C})\;\in\;\mathfrak{F}(\kappa)_{\mathtt{X}}\;\subset\;\mathtt{J}(\kappa)_{\mathtt{X}}.$$

But

$$S(\{id_X:X \to X\}) = S_{max}$$
 (cf. 12.4)

=>

$$\mathcal{S}(C) = \mathcal{S}_{\max} => \mathcal{S}_{\max} \in J(\kappa)_{X}.$$

[Note: The two descriptions of $J(\kappa)$ supplied by 12.18 are used in the proof.]

12.21 REMARK In the literature, terminology varies. For example, some authorities would say that a "Grothendieck topology" is a covering function κ which is a Grothendieck pretopology with identities whose underlying coverage is a Grothendieck coverage. Such a κ generates a "Grothendieck topology" in our sense via passage to $J(\kappa)$ (cf. 12.20).

12.22 EXAMPLE Take for κ the coverage defined in 11.14 (assuming the relevant conditions on \underline{C}) — then κ is a pretopology (cf. 11.32) with identities (...) and here $\$ \in J(\kappa)_X$ iff \$ is nonempty (cf. 12.19).

§13. SITES

Let C be a small category.

- 13.1 DEFINITION A Grothendieck topology on C is a function τ that assigns to each $X \in Ob$ C a set τ_X of sieves over X subject to the following assumptions.
 - (1) The maximal sieve $\mathcal{S}_{\max} \in \tau_{\mathbf{X}}$
 - (2) If $\mathfrak{F}\in\tau_X$ and if $f:Y\to X$ is a morphism, then $f*\mathfrak{F}\in\tau_Y$.
- (3) If § $\in \tau_X$ and if §' is a sieve over X such that f*§' $\in \tau_Y$ for all $f:Y \to X$ in §, then §' $\in \tau_X$.

[Note: Within the setting of a small category, this is just a rephrasing of the definition of "Grothendieck topology" as formulated in 12.14 (however, " κ " has been replaced by " τ " and τ_χ is a set rather than a mere conglomerate).]

- 13.2 DEFINITION A <u>site</u> is a pair (\underline{C}, τ) , where \underline{C} is a small category and τ is a Grothendieck topology on \underline{C} .
- 13.3 REMARK Suppose that we have an assignment X \rightarrow τ_{X} satisfying (1), (2) of 13.1 and for which

$$\$ \in \tau_X \& \$ \subset \$' \Rightarrow \$' \in \tau_X.$$

Then to check (3) of 13.1, it suffices to consider those \$' such that $\$' \subset \$$.

13.4 DEFINITION

- The minimal Grothendieck topology on \underline{C} is the assignment $X \to \{\$_{\max}\}$.
- The <u>maximal Grothendieck topology</u> on <u>C</u> is the assignment $X \to \{\$\}$, where \$ runs through all the sieves over X.

- 13.5 NOTATION Let $\tau_{\underline{C}}$ stand for the set of Grothendieck topologies on $\underline{C}.$
- 13.6 EXAMPLE Take $\underline{C} = \underline{1}$ then \underline{C} has two Grothendieck topologies: $\{\$_{\max}\}$ and $\{\$_{\min}, \$_{\max}\}$.

Given τ , $\tau' \in \tau_{\underline{C}'}$ write $\tau \leq \tau'$ if $\forall \ X \in Ob \ \underline{C}$, $\tau_{\underline{X}} \subset \tau_{\underline{X}}'$.

13.7 LFMMA The poset $\boldsymbol{\tau}_{\boldsymbol{C}}$ is a bounded lattice.

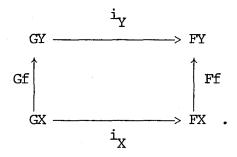
PROOF If τ , $\tau' \in \tau_{\underline{C'}}$ let $\tau \wedge \tau'$ be their set theoretical intersection and let $\tau \vee \tau'$ be the smallest Grothendieck topology containing their set theoretical union. As for 0 and 1, take 0 to be the minimal Grothendieck topology and 1 to be the maximal Grothendieck topology.

13.8 THEOREM The bounded lattice $\tau_{\underline{C}}$ is a complete Heyting algebra or, equivalently, the bounded lattice $\tau_{\underline{C}}$ is a locale.

§14. SUBFUNCTORS

Let C be a locally small category.

14.1 DEFINITION A <u>subfunctor</u> of a functor $F:\underline{C}^{OP} \to \underline{SET}$ is a functor $G:\underline{C}^{OP} \to \underline{SET}$ such that $\forall \ X \in Ob \ \underline{C}$, GX is a subset of FX and the corresponding inclusions constitute a natural transformation $G \to F$, so $\forall \ f:Y \to X$ there is a commutative diagram



[Note: There is a one-to-one correspondence between the subobjects of F and the subfunctors of F.]

14.2 LEMMA Fix an object X in \underline{C} — then there is a one-to-one correspondence between the sieves over X and the subfunctors of $h_{_{\rm X}}$ (cf. 10.8).

PROOF If \$ is a sieve over X, then the designation

$$GY = \{f: Y \rightarrow X \& f \in S\}$$

defines a subfunctor of h_X (given Z \xrightarrow{g} Y, Gg:GY \rightarrow GZ is the map f \rightarrow f \circ g). Conversely, if G is a subfunctor of h_X , then GY \subset Mor(Y,X) and

$$S = \bigcup GY$$
 Y

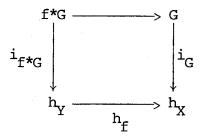
is a sieve over X.

14.3 EXAMPLE The subfunctor corresponding to $\$_{\max}$ is h_X and the subfunctor

corresponding to $\$_{\min}$ is $\emptyset_{\hat{\underline{C}}}$ (the initial object of $\hat{\underline{C}})$.

Suppose now that C is a small category — then in view of 14.2, the notion of Grothendieck topology can be reformulated.

14.4 NOTATION Given a subfunctor G of h_X and a morphism $f:Y \to X$, define f*G by the pullback square



in $\hat{\underline{C}}$ — then f*G is a subfunctor of $h_{\underline{Y}}$.

- 14.5 DEFINITION A <u>Grothendieck topology</u> on <u>C</u> is a function τ that assigns to each $X \in Ob$ <u>C</u> a set τ_X of subfunctors of h_X subject to the following assumptions.
 - (1) The subfunctor $h_X \in \tau_X$.
- (3) If $G\in\tau_X$ and if G' is a subfunctor of h_X such that $f^*\!G'\in\tau_Y$ for all $f\in GY$, then $G'\in\tau_{X^*}$
 - 14.6 LEMMA Let τ be a Grothendieck topology on C -- then

$$\texttt{G} \,\in\, \texttt{\tau}_{\texttt{X}} \,\, \texttt{\&} \,\, \texttt{G} \,\, \texttt{c} \,\, \texttt{G'} \,\, \texttt{=>} \,\, \texttt{G'} \,\, \in \,\, \texttt{\tau}_{\texttt{X}}.$$

14.7 LFMMA Let τ be a Grothendieck topology on $\underline{\textbf{C}}$ — then

$$\texttt{G,G'} \, \in \, \tau_{X} \Rightarrow \, \texttt{G} \, \cap \, \texttt{G'} \, \in \, \tau_{X}.$$

14.8 REMARK Suppose that we have an assignment $X \to \tau_{\widetilde{X}}$ satisfying (1), (2) of 14.5 and for which

$$\mathtt{G} \in \mathtt{\tau}_{\mathtt{X}} \ \mathtt{\&} \ \mathtt{G} \ \mathtt{c} \ \mathtt{G'} \ \Longrightarrow \ \mathtt{G'} \ \in \mathtt{\tau}_{\mathtt{X'}}.$$

Then to check (3) of 14.5, it suffices to consider those G' such that $G' \subset G$.

§15. SHEAVES

In what follows, all categories are assumed to be locally small for the generalities and small for the sheaf specifics.

15.1 RAPPEL A full, isomorphism closed subcategory \underline{D} of a category \underline{C} is said to be a <u>reflective</u> subcategory of \underline{C} if the inclusion $1:\underline{D} \to \underline{C}$ has a left adjoint R, a reflector for \underline{D} .

[Note: A reflective subcategory \underline{D} of a category \underline{C} is closed under the formation of limits in \underline{C} .]

Let \underline{D} be a reflective subcategory of a category \underline{C} , R a reflector for \underline{D} —then one may attach to each $X \in Ob$ \underline{C} a morphism $r_X: X \to RX$ in \underline{C} with the following property: Given any $Y \in Ob$ \underline{D} and any morphism $f: X \to Y$ in \underline{C} , there exists a unique morphism $g: RX \to Y$ in \underline{D} such that $f = g \circ r_X$.

N.B. Matters can always be arranged in such a way as to ensure that R \circ 1 = $\stackrel{\mbox{id}}{D} \cdot$

Let \underline{C} be a small category. Suppose that \underline{S} is a reflective subcategory of $\hat{\underline{C}}$. Denote the reflector by \underline{a} — then there is an adjoint pair (\underline{a}, ι) , $\iota:\underline{S} \to \hat{\underline{C}}$ the inclusion.

Assume: a preserves finite limits.

[Note: It is automatic that a preserves colimits.]

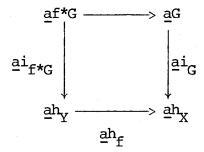
15.2 THEOREM Given $X \in Ob$ C, let τ_X be the set of those subfunctors $G \xrightarrow{i_G} h_X$ such that \underline{ai}_G is an isomorphism — then the assignment $X \to \tau_X$ is a Grothendieck

topology τ on C (in the sense of 14.5).

PROOF Since

$$\underline{\mathbf{a}}(\mathrm{id}_{\mathbf{h}_{\mathbf{X}}}) = \mathrm{id}_{\underline{\mathbf{a}}\mathbf{h}_{\mathbf{X}}}'$$

it follows that $h_X \in \tau_X$, hence (1) is satisfied. As for (2), by assumption <u>a</u> preserves finite limits, so in particular <u>a</u> preserves pullbacks, thus



is a pullback square in §. But \underline{ai}_G is an isomorphism. Therefore \underline{ai}_{f^*G} is an isomorphism, i.e., $f^*G \in \tau_V$. The verification of (3), however, is more complicated.

 \bullet Suppose that $G\in\tau_{X}$ and G is a subfunctor of G^{\prime} :

$$i_{G}:G \rightarrow h_{X}$$

$$i_{G'}:G' \rightarrow h_{X}$$

$$i_{G'}:G' \rightarrow h_{X}$$

Then

$$i_G = i_{G'} \circ i \Rightarrow \underline{a}i_G = \underline{a}i_{G'} \circ \underline{a}i.$$

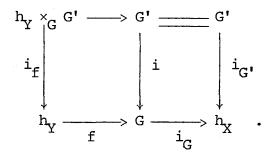
But \underline{ai}_{G} is an isomorphism, hence

$$id = \underline{ai}_{G'} \circ \underline{ai} \circ (\underline{ai}_{G})^{-1},$$

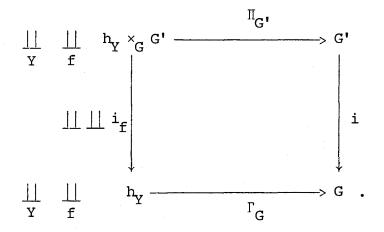
which implies that $\underline{ai}_{G'}$ is a split epimorphism. On the other hand, \underline{a} preserves monomorphisms, hence $\underline{ai}_{G'}$ is a monomorphism. Therefore $\underline{ai}_{G'}$ is an isomorphism,

i.e., $G^{\iota} \in \tau_{X}^{\iota}$.

• It remains to establish (3) under the restriction that G' is a subfunctor of G (cf. 14.8). Using the Yoneda lemma, identify each $f \in GY$ with $f \in Nat(h_Y, G)$ and display the data in the diagram



There is one such diagram for each Y and each $f \in GY$, so upon consolidation we have



Now i is an equalizer (all monomorphisms in \hat{C} are equalizers), thus $\underline{a}i$ is an equalizer (by the assumption on \underline{a}). But the assumption on G' is that \forall Y and \forall f \in GY, $\underline{a}i_f$ is an isomorphism, thus $\underline{a}i$ is an epimorphism (see 15.6 below).

And this means that <u>ai</u> is an isomorphism (in any category, a morphism which is an equalizer and an epimorphism is an isomorphism). Finally,

$$i_{G'} = i_{G} \circ i \Rightarrow \underline{a}i_{G'} = \underline{a}i_{G} \circ \underline{a}i.$$

Therefore $\underline{ai}_{G'}$ is an isomorphism, i.e., $G' \in \tau_X$.

15.3 RAPPEL Given a category C, a set U of objects in C is said to be a

a $U \in \mathcal{U}$ and a morphism $\sigma: U \to X$ such that $f \circ \sigma \neq g \circ \sigma$.

- 15.4 EXAMPLE Suppose that C is small then the h_Y (Y \in Ob C) are a separating set for \hat{C} .
- 15.5 LEMMA Let \underline{C} be a category with coproducts and let $\mathcal U$ be a separating set then $\forall \ X \in Ob \ \underline{C}$, the unique morphism

$$\begin{array}{c|c}
 & & & \Gamma_X \\
 & & \downarrow & & \text{dom } f \longrightarrow X \\
 & & \text{f } \in \text{Mor}(U,X)
\end{array}$$

such that \forall f, Γ_X \circ in = f is an epimorphism.

15.6 APPLICATION Suppose that \underline{C} is small. Working with $\hat{\underline{C}}$, take X=G in 15.5 — then

is an epimorphism.

[Note: To finish the argument that $\underline{a}i$ is an epimorphism, start with the relation

$$\Gamma_{G} \circ \coprod \coprod i_{f} = i \circ \Pi_{G}$$
.

Then

$$\underline{\mathbf{a}}\Gamma_{\mathbf{G}} \circ \underline{\mathbf{a}}(\coprod \coprod \underline{\mathbf{i}}_{\mathbf{f}}) = \underline{\mathbf{a}}\underline{\mathbf{i}} \circ \underline{\mathbf{a}}\Pi_{\mathbf{G}}.$$

Since Γ_G is an epimorphism, the same is true of $\underline{a}\Gamma_G$ (left adjoints preserve epimorphisms). And

$$\underline{\mathbf{a}}(\ \underline{\coprod}\ \underline{\coprod}\ \mathbf{i}_{\mathbf{f}}) = \underline{\coprod}\ \underline{\coprod}\ \underline{\mathbf{a}}\mathbf{i}_{\mathbf{f}}$$

is an isomorphism, call it Φ , hence

$$\underline{\mathbf{a}}\Gamma_{\mathbf{G}} = \underline{\mathbf{a}}\mathbf{i} \circ (\underline{\mathbf{a}}\Pi_{\mathbf{G}}, \circ \Phi^{-1}).$$

Therefore ai is an epimorphism.]

15.7 DEFINITION Fix a Grothendieck topology $\tau \in \tau_{\underline{C}}$ — then a presheaf $F \in Ob \ \hat{\underline{C}}$ is called a $\underline{\tau}$ -sheaf if $\forall \ X \in Ob \ \underline{C}$ and $\forall \ G \in \tau_{\underline{X}}$, the precomposition map

$$i_{G}^{*}: Nat(h_{X}, F) \rightarrow Nat(G, F)$$

is bijective.

Write $\underline{Sh}_{\tau}(\underline{C})$ for the full subcategory of $\hat{\underline{C}}$ whose objects are the τ -sheaves.

15.8 EXAMPLE Take for τ the minimal Grothendieck topology on C — then $\mathrm{Sh}_{\tau}(\underline{C}) = \hat{\underline{C}}.$

[Note: In particular, $\underline{Sh}_{T}(\underline{1}) = \hat{\underline{1}} \approx \underline{SET}$.]

15.9 EXAMPLE Take for τ the maximal Grothendieck topology on \underline{C} — then the objects of \underline{Sh}_{τ} (\underline{C}) are the final objects in $\hat{\underline{C}}$.

[First, \forall X \in Ob \underline{C} , \emptyset \rightarrow $h_{\underline{X}}$. But \emptyset is initial, thus the condition that F

be a τ -sheaf amounts to the existence for each X of a unique morphism $h_X \to F$. Meanwhile, by Yoneda, Nat $(h_X,F) \approx FX$.]

15.10 THEOREM The inclusion $\iota_{\tau}:\underline{\operatorname{Sh}}_{\tau}(\underline{\mathtt{C}})\to \hat{\underline{\mathtt{C}}}$ admits a left adjoint $\underline{\mathtt{a}}_{\tau}:\hat{\underline{\mathtt{C}}}\to \underline{\operatorname{Sh}}_{\tau}(\underline{\mathtt{C}})$ that preserves finite limits.

[Note: We can and will assume that $\mathbf{a}_{\tau} \circ \mathbf{1}_{\tau}$ is the identity.]

Various categorical generalities can then be specialized to the situation at hand.

- 15.11 DEFINITION A morphism $f:A \to B$ and an object X in a category \underline{C} are said to be <u>orthogonal</u> ($f \perp X$) if the precomposition map $f^*:Mor(B,X) \to Mor(A,X)$ is bijective.
- 15.12 RAPPEL Let \underline{D} be a reflective subcategory of a category \underline{C} , R a reflector for \underline{D} . Let $W_{\underline{D}}$ be the class of morphisms in \underline{C} rendered invertible by R.
 - Let $X \in Ob \ \underline{C}$ -- then $X \in Ob \ \underline{D}$ iff $\forall \ f \in W_{\underline{D}'}$ f $\bot X$.
 - Let $f \in Mor \ \underline{C}$ then $f \in W_{\underline{D}}$ iff $\forall \ X \in Ob \ \underline{D}$, $f \perp X$.
- 15.13 NOTATION Let $W_{_{
 m T}}$ be the class of morphisms in $\hat{\underline{\bf C}}$ rendered invertible by $\underline{\bf a}_{_{
 m T}}$.
 - 15.14 EXAMPLE If F \in Ob $\hat{\mathbf{C}}$, then F is a τ -sheaf iff \forall Ξ \in \mathcal{W}_{τ} , Ξ \bot F.

[Note: If $X \in Ob \ \underline{C}$ and if $G \in \tau_X$, then for every τ -sheaf F, $i_G \perp F$, thus $i_G \in \mathcal{W}_{\tau}$.]

- 15.16 RAPPEL Let \underline{D} be a reflective subcategory of a category \underline{C} , R a reflector for \underline{D} then the localization $W_{\underline{D}}^{-1}\underline{C}$ is equivalent to \underline{D} .
 - 15.17 APPLICATION The localization $W_{\tau}^{-1}\hat{\underline{C}}$ is equivalent to $\underline{\mathrm{Sh}}_{\tau}(\underline{C})$.
- 15.18 RAPPEL Let \underline{D} be a reflective subcategory of a finitely complete category \underline{C} , R a reflector for \underline{D} then R preserves finite limits iff $W_{\underline{D}}$ is pullback stable.
- 15.19 APPLICATION Since $\underline{a}_{\tau}: \hat{\underline{C}} \to \underline{\operatorname{Sh}}_{\tau}(\underline{C})$ preserves finite limits, it follows that W_{τ} is pullback stable.
- 15.20 EXAMPLE Take $\underline{C} = \underline{1}$, so $\hat{\underline{1}} \approx \underline{SET}$ then $\#\tau_{\underline{1}} = 2$. On the other hand, \underline{SET} has precisely 3 reflective subcategories: \underline{SET} itself, the full subcategory of final objects, and the full subcategory of final objects plus the empty set (#RX = 1 if $X \neq \emptyset$, $R\emptyset = \emptyset$). In terms of Grothendieck topologies, the first two are accounted for by 15.8 and 15.9. But the third cannot be a category of sheaves per a Grothendieck topology on $\underline{C} = \underline{1}$. To see this, note that the class of morphisms rendered invertible by R consists of all functions $f:X \to Y$ with $X \neq \emptyset$ as well as the function $\emptyset \to \emptyset$ (thus the arrows $\emptyset \to X$ ($X \neq \emptyset$) are excluded). Suppose now that Z is a nonempty set and X,Y are nonempty subsets of Z with an empty intersection. Consider the pullback square

$$\emptyset = X \cap Y \xrightarrow{\tilde{i}_X} Y$$

$$\tilde{i}_Y \downarrow \qquad \qquad \downarrow i_Y$$

$$X \xrightarrow{i_X} Z ,$$

where i_X, i_Y are the inclusions — then Ri_Y is an isomorphism but Ri_Y is not an isomorphism. Therefore the class of morphisms rendered invertible by R is not pullback stable.

15.21 NOTATION Let $F \in Ob \ \hat{\underline{C}}$ be a presheaf. Given $X \in Ob \ \underline{C}$, let $\tau_{X}(F)$ be the set of subfunctors $i_{G}: G \to h_{X}$ such that for any morphism $f: Y \to X$, the precomposition arrow

$$(i_{f*G})$$
*:Nat $(h_{Y},F) \rightarrow Nat(f*G,F)$

is bijective.

15.22 LEMMA The assignment $X \to \tau_X(F)$ is a Grothendieck topology $\tau(F)$ on \underline{C} .

 $\underline{\text{N.B.}}\ \tau(F)$ is the largest Grothendieck topology in which F is a sheaf.

15.23 SCHOLIUM For any class F of presheaves, there exists a largest Grothen-dieck topology $\tau(F)$ on C in which the $F \in F$ are sheaves.

15.24 DEFINITION The <u>canonical</u> Grothendieck topology $\tau_{\rm can}$ on C is the largest Grothendieck topology on C in which the $h_{_{\rm X}}({\rm X}\in Ob\ C)$ are sheaves.

[Note: Let $\tau \in \tau_{\underline{C}}$ — then τ is said to be <u>subcanonical</u> if the $h_X(X \in Ob\ \underline{C})$ are τ -sheaves.]

15.25 EXAMPLE Take $\underline{C} = \underline{O}(X)$, X a topological space (cf. 11.18) — then the Grothendieck topology τ on $\underline{O}(X)$ per 12.16 is the canonical Grothendieck topology, $\underline{Sh}_{T}(\underline{O}(X))$ being the traditional sheaves of sets on X, i.e., $\underline{Sh}(X)$.

§16. SHEAVES: SORITES

The category \underline{Sh}_T (C) associated with a site (C, τ) has a number of properties that will be cataloged below.

16.1 LEMMA $\underline{\operatorname{Sh}}_{\mathsf{T}}\left(\underline{\mathsf{C}}\right)$ is complete and cocomplete.

[This is because $\underline{\operatorname{Sh}}_{\mathbb{T}}(\underline{C})$ is a reflective subcategory of $\hat{\underline{C}}$ which is both complete and cocomplete. Accordingly, limits in $\underline{\operatorname{Sh}}_{\mathbb{T}}(\underline{C})$ are computed as in $\hat{\underline{C}}$ while colimits in $\underline{\operatorname{Sh}}_{\mathbb{T}}(\underline{C})$ are computed by applying $\underline{a}_{\mathbb{T}}$ to the corresponding colimts in $\hat{\underline{C}}$.]

16.2 EXAMPLE Given $\tau \in \tau_{\hbox{\bf C'}}$ define $\mathbf{0}_{\tau}$ by the rule

$$0_{\tau}(X) = \begin{bmatrix} 0 & \text{if } \emptyset & \in \tau_{X} \\ \hat{C} & & \\ 0 & \text{if } \emptyset & \notin \tau_{X}. \end{bmatrix}$$

Then $\mathbf{0}_{_{T}}$ is a $\tau\text{-sheaf}$ and, moreover, is an initial object in $\underline{Sh}_{_{T}}\left(\underline{C}\right)$.

- 16.3 LEMMA $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$ is cartesian closed.
- 16.4 LEMMA $\underline{\operatorname{Sh}}_{\mathtt{T}}\left(\underline{\mathbf{C}}\right)$ admits a subobject classifier.
- 16.5 REMARK Therefore $\underline{\operatorname{Sh}}_{\mathtt{T}}(\underline{\mathtt{C}})$ is a topos.
- 16.6 LEMMA $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$ is balanced.
- 16.7 LEMMA Every monomorphism in $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$ is an equalizer.
- [Let $\Xi:F \to G$ be a monomorphism in $\underline{\operatorname{Sh}}_{\tau}(\underline{C})$ -- then $\iota_{\tau}\Xi:\iota_{\tau}F \to \iota_{\tau}G$ is a monomorphism

in $\hat{\underline{C}}$, hence is an equalizer. But \underline{a}_{T} preserves equalizers (since it preserves finite limits).]

 $\underline{\text{N.B.}}$ Monomorphisms in $\underline{\text{Sh}}_{\mathtt{T}}\left(\underline{\mathbf{C}}\right)$ are pushout stable.

16.8 LEMMA Every epimorphism in $\underline{\operatorname{Sh}}_{\scriptscriptstyle{T}}\left(\underline{C}\right)$ is a coequalizer.

16.9 LEMMA $\underline{\operatorname{Sh}}_{\scriptscriptstyle{\mathsf{T}}}(\underline{\operatorname{C}})$ fulfills the standard conditions (cf. 11.23).

[Epimorphisms in $\underline{\operatorname{Sh}}_{\mathbb{T}}(\underline{C})$ are pullback stable (cf. 17.16) and every epimorphism in $\underline{\operatorname{Sh}}_{\mathbb{T}}(\underline{C})$ is a coequalizer (cf. 16.8).]

16.10 LEMMA In $\underline{\operatorname{Sh}}_{\mathsf{T}}\left(\underline{\mathsf{C}}\right)$, filtered colimits commute with finite limits.

16.11 RAPPEL Coproducts in $\hat{\underline{c}}$ are disjoint.

[In other words, if $F = \coprod_{i \in I} F_i$ is a coproduct of a set of presheaves F_i , then $\forall \ i \in I$, $in_i : F_i \to F$ is a monomorphism and $\forall \ i,j \in I$ ($i \neq j$), the pullback $F_i \times_F F_j$ is the initial object in $\hat{\underline{C}}$.]

16.12 LEMMA Coproducts in $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$ are disjoint.

16.13 RAPPEL Coproducts in $\hat{\underline{c}}$ are pullback stable.

[In other words, if $F = \coprod_{i \in I} F_i$ is a coproduct of a set of presheaves F_i , then for every arrow $F' \to F$,

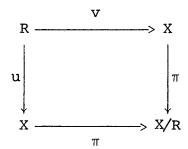
$$\coprod_{i \in I} F' \times_{F} F_{i} \approx F'.]$$

16.14 LEMMA Coproducts in $\underline{\operatorname{Sh}}_{\scriptscriptstyle{\mathbb{T}}}\left(\underline{\mathsf{C}}\right)$ are pullback stable.

16.15 DEFINITION Let C be a category which fulfills the standard conditions.

the coequalizer diagram

Then there is a commutative diagram



and a pullback square

One then says that R is effective if the canonical arrow

$$R \longrightarrow X \times_{X/R} X$$

is an isomorphism (it is always a monomorphism).

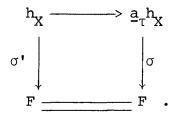
[Note: \underline{C} has <u>effective equivalence relations</u> if every equivalence relation is effective.]

16.16 LEMMA Equivalence relations in $\underline{\operatorname{Sh}}_{\mathtt{T}}\left(\underline{\mathtt{C}}\right)$ are effective.

[The usual methods apply: Equivalence relations in $\underline{\text{SET}}$ are effective, hence equivalence relations in $\underline{\text{C}}$ are effective etc.]

16.17 LEMMA The $\underline{a}_T h_X$ (X \in Ob \underline{C}) are a separating set for \underline{Sh}_T (\underline{C}).

PROOF Let $\Xi,\Xi':F \to G$ be distinct arrows in $\underline{\operatorname{Sh}}_{\mathbb{T}}(\underline{C})$ — then the claim is that $\exists X \in \operatorname{Ob} \underline{C}$ and $\sigma:\underline{a}_{\mathbb{T}}h_X \to F$ such that $\Xi \circ \sigma \neq \Xi' \circ \sigma$. But $\Xi \neq \Xi'$ implies that $\Xi_X \neq \Xi_X'$ ($\exists X \in \operatorname{Ob} \underline{C}$) which implies that $\Xi_X x \neq \Xi_X' x$ ($\exists x \in FX$). Owing to the Yoneda lemma, $FX \approx \operatorname{Nat}(h_X,F)$, so x corresponds to a $\sigma' \in \operatorname{Nat}(h_X,F)$, thus $\Xi \circ \sigma' \neq \Xi' \circ \sigma'$. Determine $\sigma:\underline{a}_{\mathbb{T}}h_X \to F$ by the diagram



Then $\Xi \circ \sigma \neq \Xi' \circ \sigma$.

N.B. All epimorphisms in $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$ are coequalizers (cf. 16.8). So, for every T -sheaf F, the epimorphism Γ_{F} of 15.5 is automatically a coequalizer. Therefore the $\underline{\mathsf{a}}_{\mathsf{T}}\mathbf{h}_{\mathsf{X}}$ (X \in Ob $\underline{\mathsf{C}}$) are a "strong" separating set for $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$.

16.18 DEFINITION Let \underline{C} be a cocomplete category and let κ be a regular cardinal — then an object $X \in Ob$ \underline{C} is κ -definite if Mor(X,—) preserves κ -filtered colimits.

16.19 LFMMA $\underline{\operatorname{Sh}}_{\mathsf{T}}(\underline{\mathsf{C}})$ is presentable.

PROOF Fix a regular cardinal κ > #Mor \underline{C} -- then \forall X \in Ob \underline{C} , $h_{\underline{X}} \in$ Ob $\underline{\hat{C}}$ is κ -definite, the contention being that \forall X \in Ob \underline{C} , $\underline{a}_{\underline{T}}h_{\underline{X}} \in$ Ob $\underline{Sh}_{\underline{T}}(\underline{C})$ is κ -definite,

which suffices. To see this, note first that a κ -filtered colimit of τ -sheaves can be computed levelwise, i.e., its κ -filtered colimit per $\hat{\underline{C}}$ is a τ -sheaf. Now fix a κ -filtered category $\underline{\underline{I}}$ and let $\Delta:\underline{\underline{I}} \to \underline{\underline{Sh}}_{\underline{I}}(\underline{\underline{C}})$ be a diagram — then

$$\begin{split} \operatorname{Nat} \left(\underline{\mathbf{a}}_{\tau} \mathbf{h}_{\mathbf{X}}, \operatorname{colim}_{\underline{\mathbf{I}}} \ \Delta_{\mathbf{i}} \right) &\approx \operatorname{Nat} \left(\underline{\mathbf{a}}_{\tau} \mathbf{h}_{\mathbf{X}}, \operatorname{colim}_{\underline{\mathbf{I}}} \ \mathbf{1}_{\tau} \Delta_{\mathbf{i}} \right) \\ &\approx \operatorname{Nat} \left(\mathbf{h}_{\mathbf{X}}, \operatorname{colim}_{\underline{\mathbf{I}}} \ \mathbf{1}_{\tau} \Delta_{\mathbf{i}} \right) \\ &\approx \operatorname{colim}_{\underline{\mathbf{I}}} \ \operatorname{Nat} \left(\mathbf{h}_{\mathbf{X}}, \mathbf{1}_{\tau} \Delta_{\mathbf{i}} \right) \\ &\approx \operatorname{colim}_{\underline{\mathbf{I}}} \ \operatorname{Nat} \left(\underline{\mathbf{a}}_{\tau} \mathbf{h}_{\mathbf{X}}, \Delta_{\mathbf{i}} \right) . \end{split}$$

- 16.20 REMARK A presentable category is necessarily wellpowered and cowell-powered.
- 16.21 DEFINITION Let \underline{E} be a topos -- then \underline{E} is said to be a <u>Grothendieck topos</u> if \underline{E} is cocomplete and has a separating set.

[Note: In general, a cocomplete topos need not admit a separating set.]

It therefore follows from 16.17 that the cocomplete topos $\underline{\mathrm{Sh}}_{\mathbb{T}}(\underline{\mathtt{C}})$ is a Grothen-dieck topos.

§17. LOCAL ISOMORPHISMS

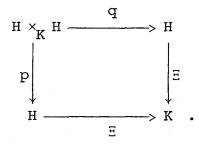
Let C be a locally small category.

- 17.1 DEFINITION Let $f: X \to Y$ be a morphism in C then a <u>decomposition</u> of k m f is a pair of arrows $X \longrightarrow M \longrightarrow Y$ such that $f = m \circ k$, where k is an epimorphism and m is a monomorphism. The decomposition (k,m) of f is said to be <u>minimal</u> (and M is said to be the <u>image</u> of f, denoted im f) if for any other factorization $X \longrightarrow N \longrightarrow Y$ of f with n a monomorphism, there is an $h: M \to N$ such that $h \circ k = \ell$ and $n \circ h = m$.
- 17.2 LEMMA Suppose that \underline{C} fulfills the standard conditions (cf. 11.23) -- then every morphism $f:X \to Y$ in \underline{C} admits a minimal decomposition $f=m \circ k$, where k is a coequalizer and m is a monomorphism, the data being unique up to isomorphism.

Let C be a small category.

17.3 RAPPEL $\hat{\underline{C}}$ fulfills the standard condtions (and is balanced).

Let $\text{H,K} \in \text{Ob } \hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat(H,K)}$. Form the pullback square



Then p and q are epimorphisms.

17.4 NOTATION δ_H :H \to H \times_K H is the canonical arrow associated with id_H, thus $p \circ \delta_H = id_H = q \circ \delta_H.$

 $\underline{\text{N.B.}}~\delta_{_{\text{\scriptsize H}}}$ is a monomorphism.

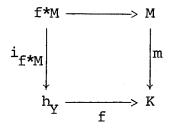
17.5 LEMMA Ξ is a monomorphism iff δ_H is an epimorphism.

[Note: Consequently, if Ξ is a monomorphism, then δ_{H} is an isomorphism.]

Fix a Grothendieck topology $\tau \in \tau_{\mathbb{C}}$.

17.6 DEFINITION Let $H,K\in Ob$ $\hat{\underline{C}}$ be presheaves and let $E\in Nat(H,K)$. Factor E per 17.2:

Then Ξ is a $\underline{\tau\text{-local epimorphism}}$ if for any $f:h_{\underline{Y}} \to K$, the subfunctor f*M of $h_{\underline{Y}}$ defined by the pullback square

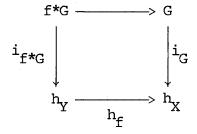


is in τ_{V} .

- 17.7 LEMMA Every epimorphism in $\hat{\underline{c}}$ is a τ -local epimorphism.
- 17.8 DEFINITION Let H,K \in Ob $\hat{\underline{C}}$ be presheaves and let $\Xi \in$ Nat(H,K) -- then Ξ is a $\underline{\tau\text{-local monomorphism}}$ if δ_H is a $\tau\text{-local epimorphism}$ (cf. 17.5).
 - 17.9 LEMMA Every monomorphism in $\hat{\underline{c}}$ is a τ -local monomorphism.

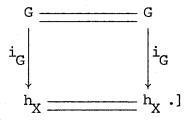
17.10 DEFINITION Let $H,K \in Ob \ \hat{C}$ be presheaves and let $\Xi \in Nat(H,K)$ -- then Ξ is a $\underline{\tau}$ -local isomorphism if Ξ is both a τ -local epimorphism and a τ -local monomorphism.

17.11 EXAMPLE If $G \in \tau_X$, then $i_G: G \to h_X$ is a τ -local isomorphism. [For any $f: Y \to X$, there is a pullback square



in $\hat{\underline{C}}$ and $f^*G \in \tau_Y$, thus i_G is a τ -local epimorphism. On the other hand, i_G is a monomorphism, hence i_G is a τ -local monomorphism (cf. 17.9).]

[Note: If G is a subfunctor of h_X and if $i_G:G\to h_X$ is a τ -local epimorphism, then $G\in\tau_X$. Proof: Take $f=id_X$ and consider



- 17.12 THEOREM \textbf{W}_{T} is the class of $\tau\text{-local}$ isomorphisms.
- 17.13 NOTATION Denote by $\underline{S}_{\underline{C}}$ the "set" of reflective subcategories \underline{S} of $\hat{\underline{C}}$ with the property that the inclusion $\iota:\underline{S}\to\hat{\underline{C}}$ has a left adjoint $\underline{a}:\hat{\underline{C}}\to\underline{S}$ that preserves finite limits.

We shall now proceed to establish the "fundamental correspondence".

17.14 THEOREM The arrows

are mutually inverse.

To dispatch the second of these, consider the composite

$$\tau_{\underline{C}} \longrightarrow \underline{s}_{\underline{C}} \longrightarrow \tau_{\underline{C}}.$$

Take a $\tau \in \tau_{\underline{C}}$ and pass to $\underline{\operatorname{Sh}}_{\tau}(\underline{C})$ — then the Grothendieck topology on \underline{C} determined by $\underline{\operatorname{Sh}}_{\tau}(\underline{C})$ via 15.2 assigns to each $X \in \operatorname{Ob} \underline{C}$ the set of those subfunctors $i_{\underline{G}}: G \to h_{X}$ such that $\underline{a}_{\tau}i_{\underline{G}}$ is an isomorphism or, equivalently, those subfunctors $i_{\underline{G}}: G \to h_{X}$ such that $i_{\underline{G}}$ is a τ -local isomorphism (cf. 17.12). But, as has been seen above, the subfunctors of h_{X} with this property are precisely the elements of τ_{X} (cf. 17.11). Therefore the composite

$${}^{\tau}_{\underline{C}} \longrightarrow {}^{\underline{S}}_{\underline{C}} \longrightarrow {}^{\tau}_{\underline{C}}$$

is the identity map.

It remains to prove that the composite

$$\underline{s}_{\underline{C}} \longrightarrow \tau_{\underline{C}} \longrightarrow \underline{s}_{\underline{C}}$$

is the identity map. So take an $\underline{S} \in \underline{S}_{\underline{C}}$, produce a Grothendieck topology τ on \underline{C} per 15.2, and pass to $\underline{Sh}_{\tau}(\underline{C})$ — then $\underline{S} \subset \underline{Sh}_{\tau}(\underline{C})$. Thus let $F \in Ob \ \underline{S}$, the claim being that $F \in \underline{Ob} \ \underline{Sh}_{\tau}(\underline{C})$ or still, that F is a τ -sheaf, or still, that $\forall \ X \in Ob \ \underline{C}$ and

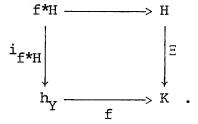
 \forall G \in $\tau_{X'}$ i_{G} \bot F, which is clear since i_{G} \in W_{τ} (cf. 15.15). To reverse matters and deduce that $\underline{\operatorname{Sh}}_{\tau}(\underline{C}) \subset \underline{S}$, one has only to show that if $\Xi:H \to K$ is a morphism in $\hat{\underline{C}}$ and if $\underline{a}\Xi$ is an isomorphism, then $\underline{a}_{\tau}\Xi$ is an isomorphism (cf. 17.17 infra). To this end, factor Ξ per 17.2:

$$\begin{array}{ccc} & & & m \\ H & \longrightarrow & M & \longrightarrow & K_{\bullet} \end{array}$$

Then <u>a</u>E = <u>a</u>m · <u>a</u>k. But <u>a</u>E is an isomorphism and <u>a</u>m is a monomorphism (<u>a</u> preserves finite limits). Therefore <u>a</u>k is a monomorphism. But <u>a</u>k is a coequalizer (<u>a</u> is a left adjoint), thus <u>a</u>k is an isomorphism (in any category, a morphism which is a monomorphism and a coequalizer is an isomorphism). And then <u>a</u>m is an isomorphism as well.

 \bullet Assume that $\underline{a}\Xi$ is an isomorphism, where Ξ is a monomorphism — then $\underline{a}_{\tau}\Xi$ is an isomorphism.

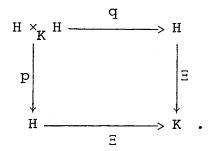
[Bearing in mind that here H = M, consider a pullback square



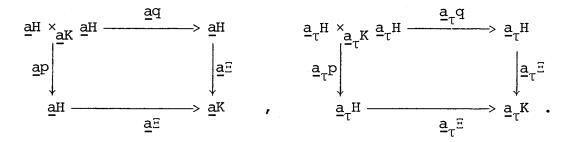
Then the assumption that $\underline{a}\Xi$ is an isomorphism implies that $\underline{a}i_{f^*H}$ is an isomorphism which in turn implies that $i_{f^*H} \in \tau_Y$. Therefore Ξ is a τ -local epimorphism or still, Ξ is a τ -local isomorphism, hence $\Xi \in W_T$ (cf. 17.12), so $\underline{a}_T\Xi$ is an isomorphism.

 \bullet Assume that $\underline{a}\Xi$ is an isomorphism, where Ξ is a coequalizer -- then $\underline{a}_{\mathsf{T}}\Xi$ is an isomorphism.

[Because $\underline{a}_{\mathsf{T}}\Xi$ is a coequalizer, to conclude that $\underline{a}_{\mathsf{T}}\Xi$ is an isomorphism, it suffices to verify that $\underline{a}_{\mathsf{T}}\Xi$ is a monomorphism. For this purpose, consider the pullback square



Then $\boldsymbol{\delta}_H$ is a monomorphism and there are pullback squares



But $\underline{a}\delta_H = \delta_{\underline{a}H}$ is an isomorphism, thus $\underline{a}_{\tau}\delta_H = \delta_{\underline{a}_{\tau}H}$ is an isomorphism (cf. supra), so $\underline{a}_{\tau}\Xi$ is a monomorphism.]

- 17.15 THEOREM Let $H,K\in Ob$ $\hat{\underline{C}}$ be presheaves and let $\Xi\in Nat(H,K)$ then $\underline{a}_{\tau}\Xi:\underline{a}_{\tau}H\to\underline{a}_{\tau}K$ is an epimorphism in $\underline{Sh}_{\tau}(\underline{C})$ iff Ξ is a τ -local epimorphism.
 - 17.16 APPLICATION The epimorphisms in $\underline{\operatorname{Sh}}_{\tau}(\underline{C})$ are pullback stable. [The class of τ -local epimorphisms is pullback stable.]
- 17.17 LEMMA Let $\underline{\mathbf{D}}_1$, $\underline{\mathbf{D}}_2$ be reflective subcategories of a category $\underline{\mathbf{C}}$. Suppose that $w_{\underline{\mathbf{D}}_2} \subset w_{\underline{\mathbf{D}}_1}$ then $\underline{\mathbf{D}}_1 \subset \underline{\mathbf{D}}_2$.

PROOF Take $X_1 \in Ob \ \underline{D}_1$. To conclude that $X_1 \in Ob \ \underline{D}_2$, it need only be shown that $\forall \ f \in \mathcal{W}_{\underline{D}_2}$, $f \perp X_1$ (cf. 15.12). But

§18. K-SHEAVES

Let C be a category.

18.1 DEFINITION Let C be a covering of $X \in Ob$ C — then a functor $F:C^{OP} \to SET$ has the sheaf property w.r.t. C if the following condition is satisfied: Given elements

$$x_q \in FY (g:Y \rightarrow X in C)$$

which are compatible in the sense that if

(i)
$$\begin{vmatrix} -h_1: Z \to \text{dom } g_1 & (g_1: Y_1 \to X \text{ in } C) \\ h_2: Z \to \text{dom } g_2 & (g_2: Y_2 \to X \text{ in } C) \end{vmatrix}$$

and

(ii)
$$g_1 \circ h_1 = g_2 \circ h_2$$

imply

(iii)
$$(Fh_1(x_{g_1}) = (Fh_2(x_{g_2}))$$

then there exists a unique $x \in FX$ such that $\forall g:Y \to X$ in C,

$$(Fg)x = x_q$$

18.2 REMARK Suppose that S is a sieve -- then elements $x_f \in FY$ (f:Y \rightarrow X in S)

are compatible iff whenever $Z \xrightarrow{g} Y \xrightarrow{} X$, there follows

$$x_{f \circ g} = (Fg)(x_{f}).$$

[Note: If C is locally small, then

say

$$S \iff G \in h_X$$

Accordingly, a compatible family corresponds to a natural transformation $G \to F$ and F has the sheaf property w.r.t. § iff every natural transformation $G \to F$ extends uniquely to a natural transformation $h_y \to F$.]

- 18.3 EXAMPLE Take $C = \{id_X: X \to X\}$ then every functor $F: \underline{C}^{OP} \to \underline{SET}$ has the sheaf property w.r.t. C.
- 18.4 LEMMA A functor $F:\underline{C}^{OP} \to \underline{SET}$ has the sheaf property w.r.t. C iff it has the sheaf property w.r.t. S(C) (cf. 12.3).
- 18.5 EXAMPLE Fix $X \in Ob \ \underline{C}$ then every functor $F:\underline{C}^{OP} \to \underline{SET}$ has the sheaf property w.r.t. $\$_{max}$ (cf. 12.4).
- 18.6 DEFINITION Suppose that κ is a covering function then a functor $F:\underline{C}^{OP} \to \underline{SET} \text{ is a } \underline{\kappa}\text{-sheaf if it has the sheaf property w.r.t. all the coverings in } \kappa.$
 - N.B. The κ -sheaves and the $\$(\kappa)$ -sheaves are one and the same.
- 18.7 REMARK Let \underline{C} be a small category and suppose that τ is a Grothendieck topology on \underline{C} then τ can be defined as in 13.1 or as in 14.5, thus there are two possible interpretations of the phrase " τ -sheaf", viz. the one above or that of 15.7. Fortunately, however, there is no ambiguity: Both are descriptions of the same entity.
 - 18.8 LEMMA If κ is a coverage and if $\kappa' \leq \kappa$, then every κ -sheaf is a κ' -sheaf.

[This is because if F is a κ -sheaf, then F has the sheaf property w.r.t. every covering that has a refinement in κ .]

18.9 APPLICATION Equivalent coverages have the same sheaves.

Write $\underline{Sh}_{\kappa}(\underline{C})$ for the full submetacategory of $[\underline{C}^{OP},\underline{SET}]$ whose objects are the κ -sheaves.

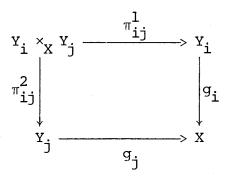
18.10 LEMMA Suppose that k is a coverage -- then

$$\frac{\operatorname{Sh}}{\kappa}(\underline{C}) = \frac{\operatorname{Sh}}{\operatorname{sat}} \kappa(\underline{C})$$
$$= \frac{\operatorname{Sh}}{\kappa}(\operatorname{sat} \kappa)(\underline{C}).$$

18.11 THEOREM Suppose that κ is a pretopology with identities -- then $J(\kappa)$ is a Grothendieck topology (cf. 12.20) and

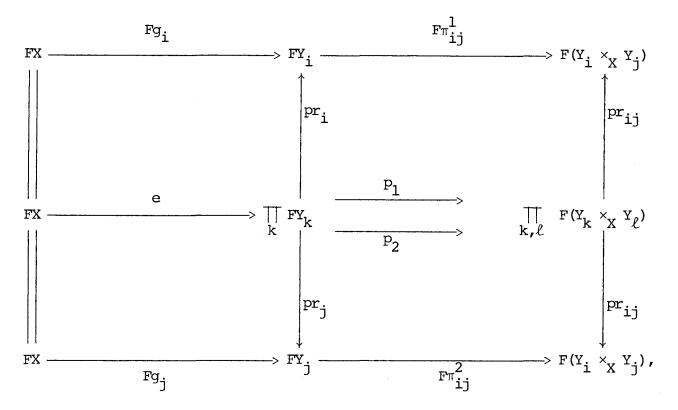
$$\underline{\operatorname{Sh}}_{\mathsf{T}(\mathsf{K})}(\underline{\mathsf{C}}) = \underline{\operatorname{Sh}}_{\mathsf{K}}(\underline{\mathsf{C}}).$$

In the presence of a size restriction and pullbacks, there is another way to formulate the sheaf property. Thus let $\mathcal C$ be a covering of $X\in Ob$ C, say $\mathcal C= G_i$ $Y_i \longrightarrow X: i\in I$, where I is set. Assume that the pullbacks



exist for all $i,j \in I$.

18.12 LEMMA Under the preceding conditions, a functor $F:\underline{C}^{OP} \to \underline{SET}$ has the sheaf property w.r.t. C iff in the diagram



e is an equalizer of p_1 and p_2 in <u>SET</u>.

- 18.13 DEFINITION Let \underline{C} be a locally small category, κ a covering function then κ is subcanonical if \forall $X \in Ob$ \underline{C} , $h_{\underline{X}}$ is a κ -sheaf.
- 18.14 EXAMPLE Assuming that \underline{C} has pullbacks, define κ by $\kappa_{\underline{X}}$ = {f}, where $f \in Ob \ \underline{C}/X$ then κ is subcanonical iff the f are coequalizers.
- 18.15 EXAMPLE Take $\underline{C} = \underline{TOP}$ then the open map coverage is subcanonical. But the open subset coverage, the open embedding coverage, and the local homeomorphism coverage are all subordinate to the open map coverage, hence they too are subcanonical (cf. 18.8).

- 18.16 EXAMPLE Take $\underline{C} = \underline{SCH}$ (cf. 0.6) and fix $X \in Ob \ \underline{C}$ (\mathcal{O}_X being understood).
- Let κ_X be comprised of the collections $\{g_i:Y_i\to X\}$ such that \forall i, g_i is an open immersion and \cup $g_i(Y_i)=X$ then κ is a Grothendieck coverage, the Zariski coverage.
- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that \forall i, g_i is étale and \cup $g_i(Y_i) = X$ then κ is a Grothendieck coverage, the <u>étale coverage</u>.
- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that \forall i, g_i is smooth and \cup $g_i(Y_i) = X$ then κ is a Grothendieck coverage, the smooth coverage.
- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that \forall i, g_i is flat + locally of finite presentation and \cup $g_i(Y_i) = X$ -- then κ is a Grothen-dieck coverage, the fppf coverage.
- 18.17 REMARK Each of these Grothendieck coverages is a Grothendieck pretopology with identities.

An open immersion is necessarily étale, an étale morphism is necessarily smooth, and a smooth morphism is necessarily flat + locally of finite presentation. Therefore the Zariski coverage is subordinate to the étale coverage which in turn is subordinate to the smooth coverage which in turn is subordinate to the fppf coverage.

[Note: If κ is the fppf coverage and if κ' is the Zariski coverage, then every κ -sheaf is a κ' -sheaf (cf. 18.8) but there are κ' -sheaves that are not κ -sheaves.]

18.18 THEOREM The fppf coverage is subcanonical.

Consequently, the Zariski coverage, the étale coverage, and the smooth coverage

are all subcanonical (cf. 18.8).

It turns out that the fppf coverage is subordinate to the so-called "fpqc coverage" (see below).

18.19 DEFINITION Let $f:X \to Y$ be a surjective morphism of schemes -- then f is <u>locally quasi-compact</u> provided that every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

18.20 EXAMPLE Let $f:X \to Y$ be a surjective morphism of schemes.

- (1) If f is quasi-compact, then f is locally quasi-compact.
- (2) If f is open, then f is locally quasi-compact.

Given a scheme X, let κ_X be comprised of the collections $\{g_i: Y_i \to X\}$ such that \forall i, g_i is flat, \cup $g_i(Y_i) = X$, and $\bigsqcup_i Y_i \to X$ is locally quasi-compact — then κ is a Grothendieck coverage, the fpqc coverage.

[Note: Like its predecessors, the fpqc coverage is a Grothendieck pretopology with identities.]

18.21 LEMMA The fppf coverage is subordinate to the fpqc coverage.

[A flat morphism locally of finite presentation is open.]

18.22 THEOREM The fpqc coverage is subcanonical.

Therefore

18.23 REMARK The coverage κ that assigns to each scheme X the collections

 $\{g_i: Y_i \to X\}$ such that $\forall i, g_i$ is flat and $\bigcup g_i(Y_i) = X$ is not subcanonical.

Returning to the generalities, let again C be a locally small category.

18.24 LEMMA Suppose that κ is a subcanonical covering function — then \forall X \in Ob C, the induced covering function $\bar{\kappa}$ on C/X is subcanonical.

18.25 EXAMPLE Take $\underline{C}=\underline{TOP}$, let κ be the open subset coverage, and fix $X\in Ob\ \underline{C}$ — then

$$\underline{\operatorname{Sh}}_{\mathsf{C}}(\underline{\mathsf{O}}(\mathsf{X})) = \underline{\operatorname{Sh}}(\mathsf{X})$$

and the inclusion $O(X) \rightarrow TOP/X$ induces an arrow

$$R: \underline{Sh}_{\kappa} (\underline{TOP}/X) \rightarrow \underline{Sh}(X)$$

of restriction. On the other hand, there is also an arrow

$$P: \underline{Sh}(X) \to \underline{Sh}_{K}(\underline{TOP}/X)$$

of prolongment and (P,R) is an adjoint pair.

§19. PRESITES

19.1 DEFINITION A <u>presite</u> is a pair (\underline{C}, κ) , where \underline{C} is a small category and κ is a covering function which is a Grothendieck pretopology with identities whose underlying coverage is a Grothendieck coverage (cf. 12.21).

Explicated:

- 19.1 DEFINITION (bis) A <u>presite</u> is a pair (\underline{C}, κ) , where \underline{C} is a small category and κ is a covering function subject to the following assumptions.
 - (1) $\forall x \in Ob \underline{C}, \{id_x: x \to x\} \in \kappa_x.$
- (2) \forall X \in Ob \underline{C} , \forall C \in $\kappa_{X'}$ \forall g:Y \rightarrow X in C, and \forall f':X' \rightarrow X, there is a pullback square

$$\begin{array}{c|c}
X' \times_{X} Y & \longrightarrow & Y \\
g' \downarrow & & \downarrow g \\
X' & \longrightarrow & X
\end{array}$$

such that the covering

belongs to $\kappa_{\chi'}$.

(3)
$$\forall$$
 X \in Ob \underline{C} , \forall C \in $\kappa_{X'}$, \forall g:Y \rightarrow X in C, and \forall C_g \in $\kappa_{Y'}$

$$\bigcup_{g \in C} g \circ C_g = \{g \circ h : g \in C \& h \in C_g\} (Z \xrightarrow{h} Y \xrightarrow{g} X)$$

belongs to $\kappa_{X^{\bullet}}$

[Note: Here, of course, it is understood that \forall X \in Ob \underline{C} , $\kappa_{\underline{X}}$ is a set of subsets of Ob C/X.]

19.2 THEOREM Suppose that (C,κ) is a presite — then

$$\underline{\operatorname{Sh}}_{J(\kappa)}(\underline{\mathbf{C}}) = \underline{\operatorname{Sh}}_{\kappa}(\underline{\mathbf{C}})$$
 (cf. 18.11)

and the elements of $\underline{\operatorname{Sh}}_{\kappa}(\underline{C})$ are characterized by the equalizer diagram figuring in 18.12.

19.3 EXAMPLE Take $\underline{C} = \underline{O}(X)$, X a topological space (cf. 11.18) and define the covering function κ as there — then the pair (\underline{C}, κ) is a presite and $J(\kappa)$ is the Grothendieck topology τ on $\underline{O}(X)$ per 12.16. And a functor $F:\underline{C}^{OP} \to \underline{SET}$ is a κ -sheaf iff for any subset $U \subset X$, any open covering $U = \bigcup_{i \in I} U_i$, and any collection $s_i \in FU_i$ ($i \in I$) such that $\forall i,j \in I$,

$$s_i | U_i \cap U_j = s_j | U_i \cap U_j$$

there exists a unique $s \in FU$ such that $s_i = s | U_i \ \forall \ i \in I$, or, equivalently, the diagram

$$FU \longrightarrow \underset{i}{\uparrow} FU_{i} \xrightarrow{\qquad \qquad } \underset{i,j}{\uparrow} F(U_{i} \cap U_{j})$$

is an equalizer diagram.

[Note: The empty covering of the empty set is admissible. Suppose that it is excluded (retaining, however, $\mathrm{id}_{g}:\emptyset\to\emptyset$) — then the result is another presite (\underline{C},κ') but now $\underline{\mathrm{Sh}}_{\mathrm{J}(\kappa')}(\underline{C})$ is $\underline{\mathrm{Sh}}(\mathrm{X}\ \ ||\ \{*\})$, the open subsets of $\mathrm{X}\ \ ||\ \{*\}$ being the empty set and any set of the form $\mathrm{U}\ \cup\ \{*\}$ with $\mathrm{U}\ \subset\ \mathrm{X}$ open. For instance, consider

the case when X is a singleton — then X \coprod {*} has two points, the underlying topological space is Sierpinski space, and $\underline{\operatorname{Sh}}_{J(\kappa^1)}(\underline{c})$ is equivalent to the arrow category $\underline{\operatorname{SET}}(\rightarrow)$.]

19.4 DEFINITION Let (\underline{C}, κ) , $(\underline{C}', \kappa')$ be presites — then a functor $\Phi:\underline{C} \to \underline{C}'$ is geometric provided the following conditions are satisfied.

$$(1) \ \forall \ X \in Ob \ \underline{C}, \ \forall \ C \in \ \kappa_{X'}$$

$$\Phi \, \circ \, \mathcal{C} \, \in \, \left(\text{sat } \kappa^{\, \bullet} \right)_{\Phi X} .$$

(2) \forall X \in Ob \underline{C} , \forall C \in $\kappa_{X'}$ \forall g:Y \rightarrow X in C, and \forall f':X' \rightarrow X, the canonical arrow

$$\Phi(X' \times_X Y) \to \Phi X' \times_{\Phi X} \Phi Y$$

is an isomorphism.

N.B. The first condition is equivalent to requiring that $\Phi \circ \mathcal{C}$ has a refinement in κ' (cf. 11.9).

- 19.5 EXAMPLE Take $\underline{C} = \underline{C}'$ then $id_{\underline{C}}$ is geometric iff $\kappa \leq \kappa'$ (cf. 11.6 (with the roles of κ and κ' reversed)).
- 19.6 NOTATION <u>PRESITE</u> is the locally small category whose objects are the presites and whose morphisms are the geometric functors.

[Note: PRESITE is a locally small large category.]

19.7 LEMMA Let (\underline{C}, κ) , $(\underline{C}', \kappa')$ be presites and suppose that $\Phi:\underline{C} \to \underline{C}'$ is a geometric functor. Let F' be a κ' -sheaf — then $F' \circ \Phi$ is a κ -sheaf.

PROOF Let $\mathcal C$ be a covering in κ — then Φ • $\mathcal C$ has a refinement in κ' , hence $\frac{g_i}{F'}$ has the sheaf property w.r.t. Φ • $\mathcal C$ (cf. 18.8). Assuming that $\mathcal C = \{Y_i \longrightarrow X: X: Y_i \longrightarrow X: X: Y_i \longrightarrow X:$

 $i \in I$, where I is a set, this means that the diagram

$$F' \Phi X \longrightarrow \prod_{i} F' \Phi Y_{i} \longrightarrow \prod_{i,j} F' (\Phi Y_{i} \times_{\Phi X} \Phi Y_{j})$$

is an equalizer diagram in SET. But

$$\Phi(Y_{i} \times_{X} Y_{j}) \approx \Phi Y_{i} \times_{\Phi X} \Phi Y_{j}$$

=>

$$F' \circ \Phi(Y_i \times_X Y_j) \approx F'(\Phi Y_i \times_{\Phi X} \Phi Y_j),$$

thus it remains only to quote 18.12.

A functor $\Phi:\underline{C}\to\underline{C}'$ determines a functor $\Phi^{OP}:\underline{C}^{OP}\to(\underline{C}')^{OP}$, from which an induced functor

$$(\Phi^{\mathrm{OP}})^*: [(\underline{\mathbf{C}}')^{\mathrm{OP}}, \underline{\mathrm{SET}}] \rightarrow [\underline{\mathbf{C}}^{\mathrm{OP}}, \underline{\mathrm{SET}}],$$

i.e.,

$$(\Phi^{OP})^*: \hat{\underline{C}}' \to \hat{\underline{C}}.$$

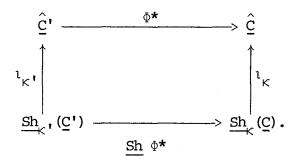
Assume now that (\underline{C}, κ) , $(\underline{C}', \kappa')$ are presites and that $\Phi:\underline{C} \to \underline{C}'$ is a geometric functor — then in 19.7, it is officially a question of

$$F' \circ \Phi^{OP} \equiv (\Phi^{OP}) *F'$$

rather than $F' \circ \Phi$. Agreeing to abbreviate $(\Phi^{OP})^*$ to Φ^* , there is an induced functor

$$\underline{\operatorname{Sh}} \, \, \Phi^* : \underline{\operatorname{Sh}}_{\mathsf{K}} \, (\underline{\mathsf{C}}') \, \to \, \underline{\operatorname{Sh}}_{\mathsf{K}} \, (\underline{\mathsf{C}})$$

and a commutative diagram



19.8 EXAMPLE Let X,Y be topological spaces and let $f:X \to Y$ be a continuous function. Define κ as in 11.18 (per X or Y) — then there are presites

$$(\underline{O}(X), \kappa) \text{ with } \underline{Sh}_{\kappa}(\underline{O}(X)) = \underline{Sh}(X)$$

$$(\underline{O}(Y), \kappa) \text{ with } \underline{Sh}_{\kappa}(\underline{O}(Y)) = \underline{Sh}(Y).$$

In addition, the functor $f^{-1}:\underline{O}(Y)\to\underline{O}(X)$ is geometric and \forall $F\in\underline{Sh}(X)$,

$$F \circ (f^{-1})^{OP} = f_{\bullet}F$$

where

$$(f_*F)V = F(f^{-1}V).$$

19.9 NOTATION Given a presite (\underline{C}, κ) , $J(\kappa)$ is a Grothendieck topology and

$$\underline{\operatorname{Sh}}_{J(\kappa)}(\underline{\mathbf{C}}) = \underline{\operatorname{Sh}}_{\kappa}(\underline{\mathbf{C}})$$
 (cf. 18.11).

Write ι_{κ} ($\equiv \iota_{J(\kappa)}$) for the inclusion $\underline{\underline{Sh}}_{\kappa}(\underline{\underline{C}}) \rightarrow \underline{\hat{\underline{C}}}$ and denote its left adjoint by $\underline{\underline{a}}_{\kappa}$ ($\equiv \underline{\underline{a}}_{J(\kappa)}$) (cf. 15.10).

Let (\underline{C}, κ) , $(\underline{C}', \kappa')$ be presites and suppose that $\Phi:\underline{C} \to \underline{C}'$ is a geometric functor — then by the theory of Kan extensions, Φ^* has a left adjoint $\Phi_!:\hat{\underline{C}} \to \hat{\underline{C}}'$.

19.10 LEMMA The composite

is a left adjoint for

$$\underline{\operatorname{Sh}} \Phi^* : \underline{\operatorname{Sh}}_{\kappa'} (\underline{\operatorname{C}}') \to \underline{\operatorname{Sh}}_{\kappa'} (\underline{\operatorname{C}}).$$

PROOF If F is a κ -sheaf and F' is a κ '-sheaf, then

$$\begin{aligned} & \operatorname{Mor}(\underline{a}_{\mathsf{K}}, \circ \Phi_{!} \circ \iota_{\mathsf{K}} F, F') \\ & \approx \operatorname{Mor}(\iota_{\mathsf{K}}, \circ \underline{a}_{\mathsf{K}}, \circ \Phi_{!} \circ \iota_{\mathsf{K}} F, \iota_{\mathsf{K}}, F') \\ & \approx \operatorname{Mor}(\underline{a}_{\mathsf{K}}, \circ \iota_{\mathsf{K}}, \circ \underline{a}_{\mathsf{K}}, \circ \Phi_{!} \circ \iota_{\mathsf{K}} F, F') \\ & \approx \operatorname{Mor}(\underline{a}_{\mathsf{K}}, \circ \Phi_{!} \circ \iota_{\mathsf{K}} F, F') \\ & \approx \operatorname{Mor}(\underline{a}_{\mathsf{K}}, \circ \Phi_{!} \circ \iota_{\mathsf{K}} F, F') \\ & \approx \operatorname{Mor}(\underline{a}_{\mathsf{K}}, \circ \Phi_{!} \circ \iota_{\mathsf{K}} F, F') \\ & \approx \operatorname{Mor}(\iota_{\mathsf{K}} F, \Phi^{*} \circ \iota_{\mathsf{K}}, F') \\ & \approx \operatorname{Mor}(\iota_{\mathsf{K}} F, \Phi^{*} \circ \iota_{\mathsf{K}}, F') \\ & \approx \operatorname{Mor}(\iota_{\mathsf{K}} F, \iota_{\mathsf{K}}, \circ \underline{Sh}, \Phi^{*} F') \\ & \approx \operatorname{Mor}(F, Sh, \Phi^{*} F') . \end{aligned}$$

19.11 REMARK The pair

$$(\underline{a}_{\kappa'}, \circ \Phi_{1} \circ \iota_{\kappa'}, \underline{Sh} \Phi^{*})$$

defines a geometric morphism

$$\underline{\operatorname{Sh}}_{\mathsf{K}}, (\underline{\mathsf{C}}') \to \underline{\operatorname{Sh}}_{\mathsf{K}}(\underline{\mathsf{C}})$$

if in addition $\underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}$ preserves finite limits.

19.12 EXAMPLE Consider the setup of 19.8. Dictionary:

$$f^{-1} \longleftrightarrow \Phi$$

$$f_{\star} \longleftrightarrow \underline{Sh} \Phi^{\star}$$

$$f^{\star} \longleftrightarrow \underline{a}_{\kappa} \circ \Phi_{!} \circ \iota_{\kappa}.$$

In traditional terminology:

[Note: The pair (f^*,f_*) defines a geometric morphism $\underline{Sh}(X) \to \underline{Sh}(Y)$.]

19.13 LEMMA There is a 2-functor

which on objects sends (\underline{C}, κ) to $\underline{\operatorname{Sh}}_{\kappa}(\underline{C})$.

N.B. It then makes sense to form

$$gro_{\underline{PRESITE}} \underline{Sh}$$
 (cf. 7.7).

19.14 EXAMPLE Take the data as in 19.8 -- then there is a functor

$$\text{TOP}^{\text{OP}} \rightarrow \text{PRESITE}$$

which on objects sends X to $(O(X),\kappa)$. From here, pass to opposites and postcompose with Sh to get a 2-functor

$$\underline{\text{TOP}} \longrightarrow \underline{\text{PRESITE}}^{\text{OP}} \xrightarrow{\underline{\text{Sh}}} 2 - CAC$$

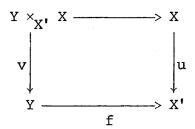
which on objects sends X to $\underline{Sh}(X)$. One may then consider its Grothendieck opconstruction

§20. INVERSE IMAGES

Let $P: \underline{E} \to \underline{B}$ be a fibration. Suppose that κ is a covering function on \underline{B} —then its inverse image $P^{-1}\kappa$ is the covering function on \underline{E} specified by the following procedure. Let $X' \in Ob \ \underline{E}$ and let $\{g: B \to PX'\} \in \kappa_{PX'}$. For each g, choose a horizontal morphism $u: X \to X'$ such that Pu = g—then the class $\{u: X \to X'\}$ is a covering of X'. One then takes for $(P^{-1}\kappa)_{X'}$ the conglomerate of all such coverings of X'.

20.1 LEMMA If κ is a coverage, then $P^{-1}\kappa$ is a coverage.

20.2 LEMMA If κ is a Grothendieck coverage, then $P^{-1}\kappa$ is a Grothendieck coverage. PROOF Referring to 11.16, take $X' \in Ob \ \underline{E}$, let $C \in (P^{-1}\kappa)_{X'}$, take $u:X \to X'$ in C, and let $f:Y \to X'$ — then the problem is to construct a pullback

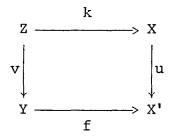


of u along f such that the covering

$$\{Y \times_{X'} X \xrightarrow{V} Y : u \in C\}$$

belongs to $(P^{-1}\kappa)_{Y}$. To this end, pass to B and form PY \times_{PX} , B per the assumption on κ :

Choose a horizontal $v:Z \to Y$ such that Pv = h, hence $PZ = PY \times_{PX}$. B, the claim being that Z is a pullback of u along f. The first step in the verification is to find a morphism $k:Z \to X$ rendering the diagram



commutative. So consider

Then

$$P(f \circ v) = Pf \circ Pv.$$

On the other hand,

$$Pu \circ pr_B = g \circ pr_B = Pf \circ h = Pf \circ Pv.$$

Accordingly, since u is horizontal, there exists a unique morphism $k:Z\to X$ such that $Pk=pr_B$ and $u\circ k=f\circ v$. There remains the universality of Z: If

Existence of ϕ Since PZ = PY \times_{PX} , B is a pullback, there is a unique $\psi: \overrightarrow{PZ} \to PZ$

such that

$$\begin{array}{|c|c|c|c|c|c|}\hline & pr_B \circ \psi & (= Pk \circ \psi) & = P\widetilde{k} \\ & h \circ \psi & (= Pv \circ \psi) & = \widetilde{Pv}. \end{array}$$

Bearing in mind that v is horizontal, consider

$$\begin{array}{c|c}
\tilde{V} & \xrightarrow{PV} \\
\downarrow \tilde{Z} & \cdot \cdot \cdot \cdot > Z & \longrightarrow Y \\
\downarrow & V & \downarrow & \downarrow & PZ & \longrightarrow PY \\
\downarrow & & & \downarrow & PV & \downarrow \\
\end{array}$$

Then

$$\overrightarrow{PV} = PV \circ \psi$$

which implies that there exists a unique morphism $\phi: \tilde{Z} \to Z$ such that $P\phi = \psi$ and $V \circ \phi = \tilde{V}$. To check that $k \circ \phi = \tilde{k}$, consider

Because u is horizontal, there is a unique morphism $\tilde{\ell}: \tilde{Z} \to X$ such that $P\ell = P\tilde{k}$ and $u \circ \tilde{\ell} = u \circ \tilde{k}$. Obviously, then, $\tilde{\ell} = \tilde{k}$. But meanwhile,

$$v \circ \phi = \tilde{v} \Rightarrow f \circ v \circ \phi = f \circ \tilde{v} = u \circ \tilde{k}.$$

I.e.:

$$u \circ k \circ \phi = u \circ \tilde{k}$$
.

And

$$P(k \circ \phi) = Pk \circ P\phi = pr_B \circ \psi = Pk.$$

Therefore $k \cdot \phi = \tilde{k}$.

<u>Uniqueness of ϕ If $\phi_1, \phi_2: \tilde{Z} \to Z$ both satisfy the requisite conditions, then</u>

- 20.3 REMARK It is not assumed that $\underline{\mathtt{B}}$ or $\underline{\mathtt{E}}$ has pullbacks but merely certain pullbacks as per the definition of Grothendieck coverage.
 - 20.4 LEMMA If κ is a pretopology, then $P^{-1}\kappa$ is a pretopology.
- 20.5 LEMMA If κ is a Grothendieck pretopology, then $P^{-1}\kappa$ is a Grothendieck pretopology.
- 20.6 LEMMA If κ is a pretopology (or a Grothendieck pretopology) with identities, then $P^{-1}\kappa$ is a pretopology (or a Grothendieck pretopology) with identities.
- 20.7 REMARK Ignoring issues of size, it follows that if (\underline{B}, κ) is a "presite", then $(\underline{E}, P^{-1}\kappa)$ is a "presite" (cf. 19.1 and 19.1 (bis)).

§21. ALGEBRAIC STRUCTURES

Let (\underline{C}, κ) be a presite.

- 21.1 LEMMA Let $F:\underline{C}^{OP} \to \underline{SET}$ be a functor -- then F is a κ -sheaf iff \forall $S \in Ob$ \underline{SET} , the presheaf $X \to Mor(S,FX)$ is a κ -sheaf.
- 21.2 DEFINITION Let \underline{A} be a locally small category with products then a functor $F:\underline{C}^{OP} \to \underline{A}$ is a κ -sheaf with values in \underline{A} if \forall $A \in Ob$ \underline{A} , the presheaf $X \to Mor(A,FX)$ is a κ -sheaf.

Write $\underline{Sh}_{\kappa}(\underline{C},\underline{A})$ for the full subcategory of $[\underline{C}^{OP},\underline{A}]$ whose objects are the κ -sheaves with values in A (thus

$$\underline{\operatorname{Sh}}_{\mathcal{L}}(\underline{C}) \equiv \underline{\operatorname{Sh}}_{\mathcal{L}}(\underline{C},\underline{\operatorname{SET}})$$
).

21.3 REMARK Let $C = \{Y_i \xrightarrow{g} X: i \in I\} \in K_{X'}$, where I is a set -- then for any functor $F: \underline{C}^{OP} \to \underline{A}$, the diagram

$$FX \longrightarrow \prod_{i} FY_{i} \longrightarrow \prod_{i,j} F(Y_{i} \times_{X} Y_{j})$$

is an equalizer diagram in \underline{A} iff $\forall~A\in Ob~\underline{A}\text{,}$ the diagram

$$\operatorname{Mor}(A,FX) \longrightarrow \prod_{i} \operatorname{Mor}(A,FY_{i}) \longrightarrow \prod_{i,j} \operatorname{Mor}(A,F(Y_{i} \times_{X} Y_{j}))$$

is an equalizer diagram in <u>SET</u>.

The central problem at this juncture is to find conditions on \underline{A} which suffice

to ensure that the inclusion

$$\iota_{\kappa}: \underline{\operatorname{Sh}}_{\kappa}(\underline{C},\underline{A}) \to [\underline{C}^{\operatorname{OP}},\underline{A}]$$

admits a left adjoint

$$\underline{\mathbf{a}}_{\kappa} : [\underline{\mathbf{C}}^{\mathsf{OP}}, \underline{\mathbf{A}}] \to \underline{\mathsf{Sh}}_{\kappa}(\underline{\mathbf{C}}, \underline{\mathbf{A}})$$

that preserves finite limits (cf. 15.10 for the case $\underline{A} = \underline{SET}$).

- Assume: \underline{A} is a <u>construct</u>, i.e., there is a faithful functor $U:\underline{A}\to \underline{SET}$ which, in addition, reflects isomorphisms.
- 21.4 EXAMPLE $\underline{\text{HTOP}}$ is not a construct. $\underline{\text{TOP}}$ is a construct but the forgetful functor U:TOP \rightarrow SET does not reflect isomorphisms.

One then imposes the following conditions on the pair (A,U).

- (1) A is complete and U is limit preserving.
- (2) $\underline{\underline{A}}$ has filtered colimits and $\underline{\underline{U}}$ is filtered colimit preserving.
- 21.5 EXAMPLE Taking for U the forgetful functor, these conditions are met by the category of abelian groups, groups, commutative rings, rings, modules over a fixed ring, vector spaces over a fixed field,

[Note: Neither coproducts nor coequalizers are preserved by U.]

- 21.6 LEMMA Let $F:\underline{C}^{OP} \to \underline{A}$ be a functor then F is a κ -sheaf with values in \underline{A} iff U \circ F is a κ -sheaf.
- 21.7 REMARK The forgetful functor $U:\underline{TOP} \to \underline{SET}$ preserves limits and colimits. On the other hand, it is not difficult to exhibit a presite (O(X), K) (cf. 19.8)

and a functor $F:\underline{O}(X)^{OP} \to \underline{TOP}$ such that $U \circ F$ is a κ -sheaf but F is not a κ -sheaf with values in TOP.

[Note: This does not contradict 21.6 (cf. 21.4).]

21.8 THEOREM The inclusion

$$\iota_{\kappa}: \underline{\operatorname{Sh}}_{\kappa}(\underline{C},\underline{A}) \to [\underline{C}^{\operatorname{OP}},\underline{A}]$$

admits a left adjoint

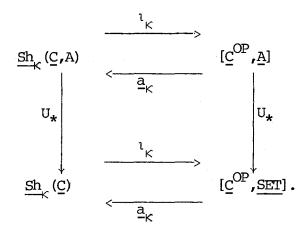
$$\underline{\mathbf{a}}_{\kappa} : [\underline{\mathbf{C}}^{\mathrm{OP}}, \underline{\mathbf{A}}] \to \underline{\mathrm{Sh}}_{\kappa} (\underline{\mathbf{C}}, \underline{\mathbf{A}})$$

that preserves finite limits.

Implicit in the proof is the fact that for any functor $F:\underline{C}^{OP} \to \underline{A}$,

$$\underline{\mathbf{a}}_{\mathsf{T}}(\mathsf{U} \, \circ \, \mathsf{F}) = \mathsf{U} \, \circ \, \underline{\mathbf{a}}_{\mathsf{T}}\mathsf{F}$$

thus there is a commutative diagram



Here \mathbf{U}_{\star} is given on objects by

$$U_{\star}F = U \circ F$$

and on morphisms by

$$(U_{\star}E)_{X} = UE_{X}$$

APPENDIX

Let \underline{C} be a category.

NOTATION <u>SIC</u> is the functor category $[\underline{\triangle}^{OP},\underline{C}]$ and a <u>simplicial object</u> in \underline{C} is an object in <u>SIC</u>.

In particular:

$$\underline{\mathtt{SISET}} = \hat{\underline{\triangle}}$$

is the category of simplicial sets.

Let \underline{C} be a small category -- then

$$\underline{\text{SI}\hat{\mathbf{C}}} = [\underline{\Delta}^{\text{OP}}, [\underline{\mathbf{C}}^{\text{OP}}, \underline{\text{SET}}]]$$

$$\approx [(\underline{\mathbf{C}} \times \underline{\Delta})^{\text{OP}}, \underline{\text{SET}}]$$

$$\approx [\underline{\mathbf{C}}^{\text{OP}}, [\underline{\Delta}^{\text{OP}}, \underline{\text{SET}}]]$$

$$= [\underline{\mathbf{C}}^{\text{OP}}, \text{SISET}],$$

the objects of the latter being termed simplicial presheaves.

Suppose that (C,κ) is a presite.

DEFINITION The objects of $\underline{\text{SISh}}_{\kappa}(\underline{\textbf{C}})$ are called $\underline{\text{simplicial }\kappa\text{-sheaves}}.$

The product $\underline{C} \times \underline{\Lambda}$ is a presite, viz.

$$\kappa_{X \times [n]} = i_n \kappa_{X'}$$

where

$$i_n:\underline{C} \to \underline{C} \times \underline{\Lambda}$$

is the inclusion

It thus makes sense to form $\underline{\operatorname{Sh}}_{\kappa}\left(\underline{C}\times\underline{\Delta}\right)$.

LEMMA We have

$$\underline{\operatorname{SISh}}_{\kappa}(\underline{C}) \approx \underline{\operatorname{Sh}}_{\kappa}(\underline{C} \times \underline{\Delta}).$$

All the basic results on presheaves and κ -sheaves of sets extend without essential change to simplicial presheaves and simplicial κ -sheaves.

N.B. It is customary to use the same symbols $\begin{bmatrix} &\underline{a}_{K}\\ && \text{for the induced adjoint} \\ && \end{bmatrix}$ pair

$$\frac{\operatorname{SI\hat{C}} \longrightarrow \operatorname{SISh}_{\mathsf{K}}(\underline{\mathsf{C}})}{\operatorname{SISh}_{\mathsf{K}}(\underline{\mathsf{C}}) \longrightarrow \operatorname{SI\hat{C}}}.$$

LEMMA $\underline{\operatorname{Sh}}_{\kappa}(\underline{\operatorname{C}},\underline{\operatorname{SISET}})$ can be identified with

$$\underline{\mathrm{SISh}}_{\kappa}(\underline{\mathrm{C}}) \approx \underline{\mathrm{Sh}}_{\kappa}(\underline{\mathrm{C}} \times \underline{\Delta}).$$

PROOF A simplicial presheaf $F:\underline{C}^{OP} \to \underline{SISET}$ determines a sequence $\{F_n\}$ of functors $F_n:\underline{C}^{OP} \to \underline{SET}$ via the prescription $F_nX = (FX)([n])$ and F is a simplicial κ -sheaf iff \forall n, F_n is a κ -sheaf. Assume now that $F:\underline{C}^{OP} \to \underline{SISET}$ is a κ -sheaf with values in \underline{SISET} — then for every simplicial set S, the presheaf $X \to Mor(S,FX)$

is a κ -sheaf. In particular: \forall n, the presheaf

$$X \rightarrow Mor(\Delta[n],FX)$$

is a k-sheaf. But

Mor(
$$\Delta[n]$$
,FX) \approx (FX)([n]) = F_nX ,

so \forall n, F_n is a κ -sheaf, i.e., F is a simplicial κ -sheaf. Conversely, if F is a simplicial κ -sheaf, then F is a κ -sheaf with values in <u>SISET</u>. To see this, given a simplicial set S, write

$$S = colim_i \Delta[n_i].$$

Then

$$\begin{aligned} & \text{Mor}(S,FX) &= & \text{Mor}(\text{colim}_{i} \ \Delta[n_{i}],FX) \\ & \approx & \text{lim}_{i} \ \text{Mor}(\Delta[n_{i}],FX) \\ & \approx & \text{lim}_{i} \ F_{n_{i}}X. \end{aligned}$$

And $\lim_{i} F_{n_{i}} \in Ob \xrightarrow{Sh} (\underline{C})$ is computed levelwise.

Let A be a locally small category with products.

22.1 NOTATION Given a topological space X, write $\underline{Sh}(X,\underline{A})$ for the category whose objects are the κ -sheaves with values in A.

[Note: Here κ is taken per 11.18, so

$$\underline{\operatorname{Sh}}(X,\underline{A}) = \underline{\operatorname{Sh}}_{\kappa}(\underline{O}(X),\underline{A}).]$$

N.B. Therefore

$$Sh(X) = Sh(X, SET)$$
.

- 22.2 EXAMPLE For any κ -sheaf F on X with values in A, FØ is a final object in A.
- 22.3 LEMMA Suppose that X is a one point space -- then the functor

$$\underline{\operatorname{Sh}}(X,\underline{A}) \xrightarrow{\operatorname{ev}} \underline{A}$$

that sends F to FX is an equivalence of categories.

- 22.4 REMARK If X is a one point space, $[O(X)^{OP}, \underline{A}]$ can be identified with the arrow category $\underline{A}(+)$. Fix a final object $*_{\underline{A}}$ in \underline{A} then the functor $\underline{A} + \underline{A}(+)$ which sends an object A to the arrow $\underline{A} \xrightarrow{!} *_{\underline{A}}$ has a left adjoint, viz. dom.
- 22.5 LEMMA Let X,Y be topological spaces and let $f:X \to Y$ be a continuous function -- then there is an induced functor

$$f_*: Sh(X,\underline{A}) \rightarrow Sh(Y,\underline{A})$$
 (cf. 19.8).

22.6 EXAMPLE Assuming that X is not empty, fix a point $x \in X$ and let $i_X: \{x\} \to X$

be the inclusion -- then there is an induced functor

$$(i_x)_*: \underline{Sh}(\{x\},\underline{A}) \to \underline{Sh}(X,\underline{A}).$$

Now choose a final object ${}^*\!\!\!\!\!{}_A$ in $\underline{{}^{\!\scriptscriptstyle A}}$, from which an induced functor

$$Sky_{x}:\underline{A} \rightarrow \underline{Sh}(X,\underline{A})$$
,

where

$$Sky_{x}(A)(U) = \begin{bmatrix} A & (x \in U) \\ & & \\ & & \\ & \underline{A} & (x \notin U). \end{bmatrix}$$

22.7 LEMMA If \underline{A} is cocomplete, then Sky_x admits a left adjoint

$$\underline{\operatorname{Sh}}(X,\underline{A}) \to \underline{A},$$

the stalk functor.

PROOF Let $\underline{O}(X)_X$ be the subcategory of $\underline{O}(X)$ whose objects are the open subsets of X containing x — then the inclusion $\iota_X : \underline{O}(X)_X \to \underline{O}(X)$ is geometric, hence there is an induced functor

$$1_{X}^{*}:\underline{\operatorname{Sh}}_{K}(\underline{O}(X),\underline{A}) \to \underline{\operatorname{Sh}}_{K}(\underline{O}(X)_{X},\underline{A}).$$

This said, consider the composite

$$\underline{\operatorname{Sh}}(X,\underline{A}) = \underline{\operatorname{Sh}}_{K}(\underline{O}(X),\underline{A}) \xrightarrow{1_{X}^{*}} \underline{\operatorname{Sh}}_{K}(\underline{O}(X)_{K},\underline{A}) \xrightarrow{\operatorname{colim}} \underline{A}.$$

22.8 DEFINITION An $\underline{\underline{A}}$ space is a pair (X, \mathcal{O}_X) , where X is a topological space and \mathcal{O}_X is a K-sheaf with values in $\underline{\underline{A}}$.

[Note: If \underline{A} is cocomplete, the stalk of \mathcal{O}_X at $x\in X$ is denoted by the symbol $\mathcal{O}_{X,X}.$]

 $\underline{\mathrm{TOP}}_{\underline{A}}$ is the category whose objects are the $\underline{\mathtt{A}}$ spaces and whose morphisms are the pairs

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y),$$

where $f:X \to Y$ is a continuous function, $f^{\#}:\mathcal{O}_{Y} \to f_{*}\mathcal{O}_{X}$ is a morphism in $\underline{\operatorname{Sh}}(Y,\underline{A})$, and $f_{*}\mathcal{O}_{X} = \mathcal{O}_{X} \circ (f^{-1})^{\operatorname{OP}}$.

[Note: The composition

$$(g,g^{\#}) \circ (f,f^{\#})$$

of

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y) \xrightarrow{(g, g^{\sharp})} (Z, O_Z)$$

has first component $g \circ f$ and second component $g_*(f^\#) \circ g^\#$ ($(g \circ f)_* = g_* \circ f_*$). And $id_{(X, \mathcal{O}_X)}$ is the arrow

$$(id_{X}, id_{\mathcal{O}_{X}})$$
 $(x, \mathcal{O}_{X}) \longrightarrow (x, \mathcal{O}_{X}).]$

N.B. Define a 2-functor $F:\underline{TOP} \to 2\text{-CAT}$ by sending X to $\underline{Sh}(X,\underline{A})$ and $f:X \to Y$ to f_* . One can then introduce $gro_{\underline{TOP}}$ F, the Grothendieck opconstruction on F. Thus its objects are the pairs (X, \mathcal{O}_X) , where \mathcal{O}_X is a κ -sheaf with values in \underline{A} , and its morphisms are the pairs

$$(x, o_x) \xrightarrow{(f, f)} (y, o_y),$$

where $f: X \to Y$ is a continuous function, $f: f_* O_X \to O_Y$ is a morphism in $\underline{Sh}(Y, \underline{A})$,

and $f_* O_X = O_X \circ (f^{-1})^{OP}$. Here

$$(g, {}^{\sharp}g) \circ (f, {}^{\sharp}f) = (g \circ f, {}^{\sharp}g \circ g_{*}({}^{\sharp}f))$$

and

$$id_{(X,O_X)} = (id_{X},id_{O_X}).$$

Conclusion: ...?

22.9 EXAMPLE Take $\underline{A} = \underline{RNG}$ (cf. 11.26) — then $\underline{TOP}_{\underline{RNG}}$ is the category of ringed spaces.

If U is an open subset of X and if $i_{\Pi}:U \to X$ is the inclusion, then

$$(i_{IJ})_*:\underline{Sh}(U,\underline{A}) \rightarrow \underline{Sh}(X,\underline{A})$$

admits a left adjoint

$$(i_{\underline{U}}) * : \underline{\operatorname{Sh}}(X, \underline{A}) \rightarrow \underline{\operatorname{Sh}}(U, \underline{A}).$$

This is true without any additional assumptions on \underline{A} . To proceed in general, however, we shall suppose that \underline{A} is complete and cocomplete and impose on \underline{A} the conditions set forth in §21, thereby ensuring that 21.8 is in force, hence that

$$f_*:\underline{Sh}(X,\underline{A}) \to \underline{Sh}(Y,\underline{A})$$

has a left adjoint

$$f^*:Sh(Y,\underline{A}) \rightarrow Sh(X,\underline{A})$$
 (cf. 19.12),

SO

$$\texttt{Mor}(\texttt{f*0}_{\texttt{Y}}, \texttt{0}_{\texttt{X}}) \approx \texttt{Mor}(\texttt{0}_{\texttt{Y}}, \texttt{f}_{\texttt{*}} \texttt{0}_{\texttt{X}}) \,,$$

with arrows of adjunction

$$\begin{vmatrix} \mu_{\mathcal{O}_{\mathbf{Y}}} : \mathcal{O}_{\mathbf{Y}} & \longrightarrow & f_{\star} f^{\star} \mathcal{O}_{\mathbf{Y}} \\ \nu_{\mathcal{O}_{\mathbf{X}}} : f^{\star} f_{\star} \mathcal{O}_{\mathbf{X}} & \longrightarrow & \mathcal{O}_{\mathbf{X}}. \end{vmatrix}$$

22.10 NOTATION Let $P_{\underline{A}}: \underline{TOP}_{\underline{A}} \to \underline{TOP}$ be the functor that sends $(X, 0_X)$ to X and $(f, f^{\#})$ to f.

22.11 LEMMA $\mathbf{P}_{\mathbf{A}}$ is a fibration.

PROOF Given $(Y, O_{\underline{Y}})$ and $f: X \to Y$, the morphism

$$(\mathtt{f},\mu_{\mathcal{O}_{\mathbf{Y}}}):(\mathtt{X},\mathtt{f}\star\mathcal{O}_{\mathbf{Y}})\to(\mathtt{Y},\mathcal{O}_{\mathbf{Y}})$$

is horizontal.

22.12 EXAMPLE Take X = U, Y = X, f =
$$i_U$$
 -- then $i_U^* O_X = O_X | U$ and
$$(i_U, \mu_{O_X}) : (U, O_X | U) \to (X, O_X)$$

is horizontal. Here

$$\mu_{\mathcal{O}_{X}}:\mathcal{O}_{X} \to \mathbf{i}_{\star}(\mathcal{O}_{X}|U)$$

at an open subset V < X is computed by

$$\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$$

per $U \cap V \rightarrow V$.

Let

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

be a morphism of \underline{A} spaces — then $f^{\sharp}: \mathcal{O}_{\underline{Y}} \to f_{\star} \mathcal{O}_{\underline{X}}$ is a morphism in $\underline{\operatorname{Sh}}(\underline{Y},\underline{A})$, thus corresponds to a morphism $f_{\sharp}: f^{\star} \mathcal{O}_{\underline{Y}} \to \mathcal{O}_{\underline{X}}$ in $\underline{\operatorname{Sh}}(\underline{X},\underline{A})$ under the identification

$$\mathsf{Mor}(\mathsf{f}^*\mathcal{O}_{\mathsf{Y}},\mathcal{O}_{\mathsf{X}}) \approx \mathsf{Mor}(\mathcal{O}_{\mathsf{Y}},\mathsf{f}_*\mathcal{O}_{\mathsf{X}}).$$

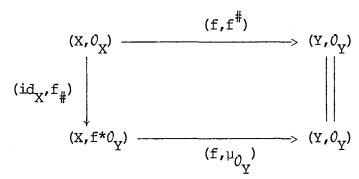
[Note: The composite

$$f*O_{Y} \xrightarrow{f*(f^{\#})} f*f_{*}O_{X} \xrightarrow{\lor O_{X}} O_{X}$$

is $f_{\#}$. Observe too that

$$(id_{X}, f_{\#}): (X, O_{X}) \longrightarrow (X, f*O_{Y})$$

is a morphism of <u>A</u> spaces: $f*0_Y \xrightarrow{f_\#} (id_X)_*0_X = 0_X$ and the diagram



in $\underline{\text{TOP}}_{\underline{A}}$ commutes.]

Consequently, at the level of stalks, $\forall \ x \in X$, there is a morphism

$$(f_{\#})_{X}: (f * O_{Y})_{X} \rightarrow O_{X,X}$$

in A.

22.13 LEMMA Fix $x \in X$ — then the stalk functor at f(x) is the composition $(i_y)^* \circ f^*.$

[The functor $(i_x)^* \circ f^*$ is a left adjoint for $f_* \circ (i_x)_* = (f \circ i_x)_* = (i_{f(x)})_{*}$.]

[Note: Technically,

$$\frac{(i_{x})^{*}}{\underline{Sh}(X,\underline{A})} \xrightarrow{Sh}(\{x\},\underline{A})$$

so "taking the stalk at x" is really $(i_{\chi})^*$ modulo the equivalence

$$\underline{\operatorname{Sh}}(\{x\},\underline{A}) \longrightarrow \underline{A} \quad (\text{cf. 22.3).}]$$

22.14 APPLICATION $\forall x \in X$,

$$O_{Y,f(x)} = (i_x) * (f*O_Y) = (f*O_Y)_x.$$

In particular:

$$(f_* O_X)_{f(x)} = (f * f_* O_X)_X$$

Fix a one point space * and consider X \longrightarrow * -- then

$$!_*: \underline{Sh}(X,\underline{A}) \rightarrow \underline{Sh}(*,\underline{A}).$$

Now postcompose !* with the equivalence $\underline{\operatorname{Sh}}(\star,\underline{\operatorname{A}}) \xrightarrow{\quad \text{ev} \quad} \underline{\operatorname{A}}$ of 22.3 to get a functor

$$\Gamma:\underline{\operatorname{Sh}}(X,\underline{A})\to\underline{A},$$

the global section functor:

$$\Gamma F = FX.$$

[Note: If

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

is a morphism of A spaces, then

$$\Gamma O_{Y} = O_{Y}(Y) \xrightarrow{f_{Y}^{\#}} (f_{*}O_{X}) (Y)$$

$$= o_{X}(f^{-1}Y)$$
$$= o_{X}(X) = Fo_{X}$$

is a morphism in A.

- 22.15 LEMMA The global section functor Γ is the restriction to $\underline{Sh}(X,\underline{A})$ of lim: $[\underline{O}(X)^{OP},\underline{A}]$.
- 22.16 RAPPEL The functor

$$\lim: [O(X)^{OP}, A] \rightarrow A$$

is a right adjoint for the constant diagram functor

$$K: \underline{A} \rightarrow [\underline{O}(X)^{OP}, \underline{A}].$$

Display the data:

$$\underline{\underline{A}} \xrightarrow{K} [\underline{O}(X)^{OP}, \underline{\underline{A}}] \xrightarrow{\underline{a}_{K}} \underline{\underline{sh}}(X, \underline{\underline{A}}).$$

$$\underline{\underline{A}} \xrightarrow{1_{K}} \underline{\underline{h}}(X, \underline{\underline{A}}).$$

Then a left adjoint for

$$\Gamma = \lim \circ \iota_{\kappa}$$

is

$$\Delta = \underline{\mathbf{a}}_{\mathsf{K}} \, \circ \, \mathsf{K}.$$

22.17 EXAMPLE Let A be a commutative ring with unit. Consider the ringed space (Spec A, \mathcal{O}_{A}) — then

$$\Gamma O_{A} = O_{A}(\text{Spec A}) \approx A.$$

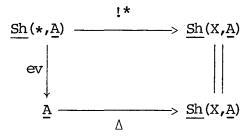
[Note: Here $\theta_A \equiv \theta_{\text{Spec A}}$ is the structure sheaf of Spec A.]

22.18 REMARK Spec $A = \emptyset$ iff $A = \{0\}$ (a zero ring). Of course, $\{0\}$ is a final object in RNG and

$$0_{\{0\}}$$
 Spec $\{0\} = 0_{\{0\}}$ $\emptyset = \{0\}$

in agreement with 22.2.

22.19 LEMMA The diagram



commutes up to isomorphism:

$$!* \approx \Delta \circ ev.$$

PROOF For any $\theta_{\mathbf{x}}$ and for any $\theta_{\mathbf{x}'}$

$$\begin{split} & \operatorname{Mor}(! * \mathcal{O}_{\star}, \mathcal{O}_{X}) \approx \operatorname{Mor}(\mathcal{O}_{\star}, !_{\star} \mathcal{O}_{X}) \\ & \approx \operatorname{Mor}(\mathcal{O}_{\star} *, (!_{\star} \mathcal{O}_{X}) \ (*)) \\ & \approx \operatorname{Mor}(\operatorname{ev} \ \mathcal{O}_{\star}, \mathcal{O}_{X} (X)) \\ & \approx \operatorname{Mor}(\operatorname{ev} \ \mathcal{O}_{\star}, \Gamma \mathcal{O}_{X}) \\ & \approx \operatorname{Mor}(\Delta \circ \operatorname{ev} \ \mathcal{O}_{\star}, \mathcal{O}_{X}). \end{split}$$

§23. LOCALLY RINGED SPACES

Let C be a category.

23.1 DEFINITION A subcategory \underline{D} of \underline{C} is said to be <u>replete</u> if for any object X in \underline{D} and for any isomorphism $f:X \to Y$ in \underline{C} , both Y and f are in \underline{D} .

[Note: If D is a full subcategory of C, then the term is <u>isomorphism closed</u>.

E.g.: Reflective subcategories are isomorphism closed.]

- 23.2 EXAMPLE Let LOC-RNG be the subcategory of RNG whose objects are the local rings and whose morphisms are the local homomorphisms then LOC-RNG is a replete (nonfull) subcategory of RNG.
- 23.3 DEFINITION Let $\underline{C},\underline{C}'$ be categories then a functor $F:\underline{C} \to \underline{C}'$ is said to be <u>replete</u> if it has the isomorphism lifting property (cf. 1.23), i.e., if \forall isomorphism $\psi:FX \to X'$ in \underline{C}' , \exists an isomorphism $\phi:X \to Y$ in \underline{C} such that $F\phi = \psi$ (so FY=X').

[Note: One can thus say that a subcategory \underline{D} of \underline{C} is replete provided the inclusion functor $\underline{D} \to \underline{C}$ is replete.]

- 23.4 EXAMPLE A fibration P:E \rightarrow B is replete (cf. 4.23).
- 23.5 LEMMA Let $F: (\underline{F}, P) \to (\underline{F}', P')$ be a morphism in CAT/\underline{B} , where $P: \underline{F} \to \underline{B}$, $P': \underline{F}' \to \underline{B}$ are fibrations then F is replete iff $\forall B \in Ob \ \underline{B}$, the functor $F_B: \underline{F}_B \to \underline{F}_B$ is replete.
- 23.6 REMARK The fiberwise condition on F amounts to the assertion that if $\psi:FX \to X'$ is a vertical isomorphism in \underline{E}' , then there exists a vertical isomorphism $\phi:X \to Y$ in E such that $F\phi = \psi$ (so FY = X').

23.7 DEFINITION A ringed space (X,0_X) is a <u>locally ringed space</u> if each stalk $\theta_{\rm X,x}$ is a local ring.

[Note: $m_{X,x}$ is the maximal ideal of $0_{X,x}$ and $\kappa(x) = 0_{X,x}/m_{X,x}$ is the residue field of $0_{X,x}$.]

23.8 REMARK Consider the pair $(\emptyset, \mathcal{O}_{g})$, where $\mathcal{O}_{g}\emptyset = \{0\}$ (a zero ring) (cf. 22.18) — then there is no stalk and the local ring condition is vacuous, so $(\emptyset, \mathcal{O}_{g})$ is a locally ringed space.

[Note: Zero rings are not local rings.]

Let $(\mathbf{X}, \mathbf{0}_{\mathbf{X}})$, $(\mathbf{Y}, \mathbf{0}_{\mathbf{Y}})$ be locally ringed spaces. Suppose that

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

is a morphism of ringed spaces — then (f,f^{\sharp}) is a morphism of locally ringed spaces if $\forall \ x \in X$, the ring homomorphism

$$(f_{\sharp})_{x}: \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$$

is local.

23.9 NOTATION Let

$\underline{\text{LOC-TOP}}_{RNG}$

be the subcategory of $\frac{\text{TOP}_{RNG}}{\text{end}}$ (cf. 22.9) whose objects are the locally ringed spaces and whose morphisms are the morphisms of locally ringed spaces.

[Note: To verify closure under composition, recall that

$$(X, \mathcal{O}_{X}) \xrightarrow{(f, f^{\sharp})} (Y, \mathcal{O}_{Y}) \xrightarrow{(g, g^{\sharp})} (Z, \mathcal{O}_{Z})$$

has first component $g \circ f$ and second component $g_*(f^{\sharp}) \circ g^{\sharp}$. And here

$$(g \circ f) * \approx f * \circ g * (...)$$

while

$$(g_*(f^{\sharp}) \circ g^{\sharp})_{\sharp} = f_{\sharp} \circ f^*(g_{\sharp}),$$

i.e.,

$$\texttt{f*g*0}_{\textbf{Z}} \xrightarrow{\texttt{f*}(\texttt{g}_{\#})} \texttt{f*0}_{\textbf{Y}} \xrightarrow{\texttt{f}_{\#}} \textit{0}_{\textbf{X}}.$$

So, \forall $x \in X$, the stalk homomorphism

$$((g_*(f^{\sharp}) \circ g^{\sharp})_{\sharp})_{x}$$

is the arrow

$$(f_{\#} \circ f^*(g_{\#}))_{x}$$

which when explicated is the composition

$$O_{Z,g} \circ f(x) \xrightarrow{(g_{\#})_{f(x)}} O_{Y,f(x)} \xrightarrow{(f_{\#})_{x}} O_{X,x}$$

of two local homomorphisms, thus is a local homomorphism.]

The functor

$$P_{\underline{RNG}} : \underline{TOP}_{\underline{RNG}} \longrightarrow \underline{TOP}$$
 (cf. 22.10)

restricts to

call it LOC-P_{RNG}.

23.10 LEMMA $\underline{\text{LOC}}\text{-P}_{\mbox{RNG}}$ is a fibration.

PROOF In the notation of the proof of 22.11, if $(Y, 0_Y)$ is a locally ringed space, then so is $(X, f*0_Y)$ $(\forall x \in X, (f*0_Y)_X = 0_{Y, f(X)})$. Moreover,

$$(f, \mu_{\mathcal{O}_{\mathbf{Y}}}) : (\mathbf{X}, f * \mathcal{O}_{\mathbf{Y}}) \rightarrow (\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$$

is a morphism of locally ringed spaces:

$$\mu_{\mathcal{O}_{\mathbf{Y}}} \in \text{Mor}(\mathcal{O}_{\mathbf{Y}}, \mathbf{f_{*}f^{*}\mathcal{O}_{\mathbf{Y}}})$$

=>

$$(\mu_{\mathcal{O}_{\mathbf{Y}}})_{\#} \in Mor(f*\mathcal{O}_{\mathbf{Y}}f*\mathcal{O}_{\mathbf{Y}})$$

or still,

$$(\mu_{\mathcal{O}_{Y}}) = id_{f*\mathcal{O}_{Y}}.$$

In addition, it is horizontal when viewed from the perspective of $\frac{TOP_{RNG}}{}$. Consider now a setup

$$\frac{(h,h^{\sharp})}{(z,\mathcal{O}_{z}) \cdot \cdot \cdot \cdot > (x,f^{*}\mathcal{O}_{y})} \xrightarrow{(f,\mu_{\mathcal{O}_{y}})} (y,\mathcal{O}_{y})^{i} , \stackrel{i}{z} \xrightarrow{g} x \xrightarrow{f} Y^{i} (h = f \circ g),$$

where (h,h^{\sharp}) is a morphism of locally ringed spaces — then there is a unique filler

$$(g,g^{\sharp}):(Z,\mathcal{O}_{Z}) \rightarrow (X,f*\mathcal{O}_{Y})$$

in $\overline{\text{TOP}}_{RNG}$ such that

$$(f, \mu_{O_{\mathbf{Y}}}) \circ (g, g^{\sharp}) = (h, h^{\sharp}),$$

the claim being that (g,g^{\sharp}) is a morphism of locally ringed spaces. To begin with

$$g^{\sharp}:f*O_{Y} \rightarrow g_{*}O_{Z}.$$

On the other hand,

$$h^{\sharp}: \mathcal{O}_{Y} \to h_{\star} \mathcal{O}_{Z} = (f \circ g)_{\star} \mathcal{O}_{Z}$$

$$= f_{\star} g_{\star} \mathcal{O}_{Z}.$$

And

$$\text{Mor}(f^*\theta_{Y'}g_*\theta_{Z}) \approx \text{Mor}(\theta_{Y'}f_*g_*\theta_{Z}),$$

hence under this identification,

$$h^{\#} \in Mor(\mathcal{O}_{V}, f_{*}g_{*}\mathcal{O}_{Z})$$

corresponds to an element

$$h_{\#f} \in Mor(f*\mathcal{O}_{Y}, g_*\mathcal{O}_{Z})$$

which, in fact, is precisely $g^{\#}$ (since $f_{*}(h_{\#f}) \circ \mu_{\mathcal{O}_{\underline{Y}}} = h^{\#}$). Accordingly, to ascertain that $\forall \ z \in Z$, $(g_{\#})_{z}$ is local, it suffices to consider $(h_{\#f}, \#g)_{z}$:

$$\begin{array}{l} \mathbf{h}_{\#\mathbf{f}} \in \operatorname{Mor}(\mathbf{f}^{*}\mathcal{O}_{\mathbf{Y}}, \mathbf{g}_{*}\mathcal{O}_{\mathbf{Z}}) \\ < & \longrightarrow \\ \mathbf{h}_{\#\mathbf{f}}, \#\mathbf{g} \in \operatorname{Mor}(\mathbf{g}^{*}\mathbf{f}^{*}\mathcal{O}_{\mathbf{Y}}, \mathcal{O}_{\mathbf{Z}}) \\ \\ & \approx \operatorname{Mor}((\mathbf{f} \circ \mathbf{g})^{*}\mathcal{O}_{\mathbf{Y}}, \mathcal{O}_{\mathbf{Z}}) \\ \\ & \approx \operatorname{Mor}(\mathbf{h}^{*}\mathcal{O}_{\mathbf{Y}}, \mathcal{O}_{\mathbf{Z}}) \end{array}.$$

But

Mor
$$(h^*\theta_Y, \theta_Z) \approx Mor (\theta_Y, h_*\theta_Z)$$
.

Therefore

$$h_{\#f,\#g}$$
 " = " $h_{\#}$.

And, \forall z \in Z, $(h_{\#})_z$ is, by hypothesis, local.

N.B. The pair

$$(\underline{\text{LOC-TOP}_{RNG}}, \underline{\text{LOC-P}_{RNG}})$$

and the pair

$$(\underline{\text{TOP}}_{RNG}, \underline{P}_{RNG})$$

are objects of

and the inclusion functor

$$\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}} \rightarrow \underline{\text{TOP}}_{\underline{\text{RNG}}}$$

is horizontal.

[Suppose that

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

is horizontal in $\underline{\text{LOC-TOP}}_{RNG}$. To see that it is horizontal in $\underline{\text{TOP}}_{RNG}$, introduce

$$(\mathtt{f},\mu_{\mathcal{O}_{\mathbf{Y}}}):(\mathtt{X},\mathtt{f}^{\star}\mathcal{O}_{\mathbf{Y}}) \ \rightarrow \ (\mathtt{Y},\mathcal{O}_{\mathbf{Y}})$$

which is horizontal in $\frac{\text{TOP}}{\text{RNG}}$ — then there is a vertical isomorphism

$$v: (X, O_X) \rightarrow (X, f*O_Y)$$

and a commutative diagram

SO

$$(f,f^{\sharp}) = (f,\mu_{O_{V}}) \circ v$$

is horizontal (cf. 4.20 and 4.21).]

23.11 LEMMA $\underline{\text{LOC-TOP}_{\underline{RNG}}}$ is a replete (nonfull) subcategory of $\underline{\text{TOP}_{\underline{RNG}}}$. [This is an application of 23.5 (and 23.6). Thus let

$$(id_X, (id_X)^{\sharp}): (X, \mathcal{O}_X) \rightarrow (X, \mathcal{O}_X')$$

be a vertical isomorphism in $\underline{\text{TOP}}_{\underline{\text{RNG}}}$, where (X, \mathcal{O}_X) is in $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ — then (X, \mathcal{O}_X) is necessarily a locally ringed space and $(\mathrm{id}_X, (\mathrm{id}_X)^\#)$ is a morphism of locally ringed spaces.]

[Note: It follows that the inclusion functor

$$\underline{\text{TOC-TOP}}_{\text{RNG}} \rightarrow \underline{\text{TOP}}_{\text{RNG}}$$

reflects isomorphisms.]

23.12 REMARK Suppose that $(Y, \mathcal{O}_{\underline{Y}})$ is a locally ringed space. Let $f: X \to Y$ be a continuous function and let

$$(f,f^{\sharp}):(X,\mathcal{O}_{X}) \rightarrow (Y,\mathcal{O}_{Y})$$

be a horizontal morphism in $\frac{\text{TOP}_{\underline{RNG}}}{\text{TNG}}$ — then (X, \mathcal{O}_{X}) is a locally ringed space and (f, f^{\sharp}) is a morphism of locally ringed spaces.

[First choose a horizontal morphism

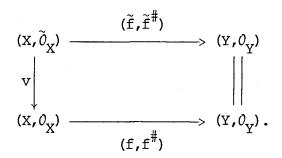
$$(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}^{\sharp}) : (\mathbf{X}, \tilde{\mathbf{O}}_{\mathbf{X}}) \rightarrow (\mathbf{Y}, \mathbf{O}_{\mathbf{Y}}) \quad (\tilde{\mathbf{f}} = \mathbf{f})$$

in $\underline{\text{LOC-TOP}_{\underline{\text{RNG}}}}$ -- then $(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}^{\sharp})$ is a horizontal morphism in $\underline{\text{TOP}_{\underline{\text{RNG}}}}$, so there is a

vertical isomorphism

$$v: (X, \tilde{\mathcal{O}}_X) \rightarrow (X, \mathcal{O}_X)$$

and a commutative diagram



Since $\underline{\text{LOC-TOP}_{\underline{RNG}}}$ is a replete subcategory of $\underline{\text{TOP}_{\underline{RNG}}}$, both (X, \mathcal{O}_X) and v are in $\underline{\text{LOC-TOP}_{\underline{RNG}}}$. Finally,

$$(f,f^{\sharp}) \circ v = (\tilde{f},\tilde{f}^{\sharp})$$

=> $(f,f^{\sharp}) = (\tilde{f},\tilde{f}^{\sharp}) \circ v^{-1}$,

hence (f,f^{\sharp}) is a morphism of locally ringed spaces (and, as such, is horizontal).]

23.13 DEFINITION An <u>affine scheme</u> is a locally ringed space which is isomorphic as a locally ringed space to (Spec A, O_A) ($O_A \equiv O_{Spec A}$) for some A \in Ob RNG (cf. 22.17).

[Note: A ringed space which is isomorphic as a ringed space to a (Spec A, O_A) is automatically a locally ringed space and the isomorphism is one of locally ringed spaces.]

- 23.14 NOTATION $\underline{\text{AFF-SCH}}$ is the full subcategory of $\underline{\text{LOC-TOP}_{\underline{\text{RNG}}}}$ whose objects are the affine schemes.
 - 23.15 REMARK The category AFF-SCH has finite products and pullbacks, hence is

finitely complete.

23.16 THEOREM The functor

$$(Spec, 0): RNG^{OP} \rightarrow AFF-SCH$$

that sends A to (Spec A, $\mathcal{O}_{\mathbf{A}}$) is an equivalence of categories.

N.B. We shall also view (Spec,0) as a fully faithful functor

$$RNG^{OP} \rightarrow LOC-TOP_{RNG}$$

Let

$$\Gamma: \underline{\text{LOC-TOP}_{\text{RNG}}} \rightarrow \underline{\text{RNG}}^{\text{OP}}$$

be the functor defined on objects (X, $\theta_{\rm X}$) by

$$\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$$

and on morphisms

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

by

$$f_{Y}^{\#}: \mathcal{O}_{Y}(Y) \rightarrow \mathcal{O}_{X}(X)$$
.

23.17 THEOREM The functor Γ is a left adjoint for the functor (Spec,0):

$$\texttt{Mor}(\Gamma(X,\mathcal{O}_{X}),\mathbb{A}) \approx \texttt{Mor}((X,\mathcal{O}_{X}),(\texttt{Spec}\ \mathbb{A},\mathcal{O}_{\underline{A}})).$$

23.18 APPLICATION (Spec Z, Q_Z) is a final object in $\underline{LOC-TOP_{RNG}}$.

[Indeed,

$$Mor(\Gamma(X, O_X), Z)$$
 in RNG^{OP}

is

Mor(
$$Z,\Gamma(X,O_X)$$
) in RNG.]

- 23.19 DEFINITION A scheme is a locally ringed space with the property that every point has an open neighborhood which is an affine scheme.
- 23.20 NOTATION $\underline{\text{SCH}}$ is the full subcategory of $\underline{\text{LOC-TOP}_{\underline{\text{RNG}}}}$ whose objects are the schemes (cf. 0.6).

[Note: AFF-SCH is a full subcategory of SCH.]

23.21 REMARK The category \underline{SCH} has finite products and pullbacks, hence is finitely complete.

[Note: SCH does not have arbitrary products, hence is not complete. Consider, for example $\prod_{1}^{\infty} P_{\mathbb{C}}^{1}$.]

 $\underline{\text{N.B.}}$ If A is a zero ring, then Spec A is an initial object in $\underline{\text{SCH}}$ whereas Spec Z is a final object in SCH.

When dealing with schemes, one sometimes says "let X be a scheme" rather than "let (X, \mathcal{O}_X) be a scheme."

- 23.22 DEFINITION Let X be a scheme -- then an open subset $U \subset X$ is an <u>affine</u> open subset of X if U is an affine scheme.
 - 23.23 LEMMA The affine open subsets of a scheme X constitute a basis for the

topology on X.

[Note: Therefore every open subset of X is a scheme.]

23.24 REMARK The intersection of two affine open subsets of X is open but it need not be affine open.

[Note: Let X be a scheme.

- X is <u>semi-separated</u> if for each pair $U,V\subset X$ of affine opens the intersection $U\cap V$ is affine open.
- X is quasi-separated if for each pair $U,V\subset X$ of affine opens the intersection $U\cap V$ is a finite union of affine opens.

One has

separated => semi-separated => quasi-separated.

Every affine scheme is separated.]

23.25 LEMMA The underlying topology on a scheme X is locally quasi-compact. [Recall that \forall A \in Ob <u>RNG</u>, Spec A is quasi-compact (but rarely Hausdorff or even T_1). On the other hand, an open subset of Spec A is not necessarily quasi-compact (although this will be the case if, e.g., A is noetherian).]

23.26 DEFINITION Let I be a set.

- ullet Given $i \in I$, let X_i be a scheme.
- ullet Given i,j \in I, let $\mathtt{U}_{\mbox{ij}}$ \subset $\mathtt{X}_{\mbox{i}}$ be an open subset and let

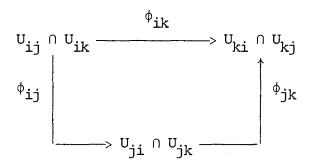
$$\phi_{ij}^{U_{ij}} \rightarrow U_{ji}$$

be an isomorphism of schemes (take $U_{ii} = X_i$ and $\phi_{ii} = id_{X_i}$).

Given i,j,k ∈ I, assume that

$$\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$

and that the diagram



commutes.

Then the collection

$$(\mathtt{I},(\mathtt{X}_{\mathtt{i}}\!:\!\mathtt{i}\in\mathtt{I})\ ,\ (\mathtt{U}_{\mathtt{i}\bar{\mathtt{j}}}\!:\!\mathtt{i},\mathtt{j}\in\mathtt{I})\ ,\ (\phi_{\mathtt{i}\bar{\mathtt{j}}}\!:\!\mathtt{i},\mathtt{j}\in\mathtt{I}))$$

is called glueing data.

23.27 THEOREM Given glueing data, there exists a scheme X, open subschemes $U_i \subset X$, with $X = \bigcup_{i \in I} U_i$, and isomorphisms $\phi_i : X_i \to U_i$ of schemes such that

(1)
$$\phi_{\mathbf{i}}(U_{\mathbf{i}\dot{\mathbf{j}}}) = U_{\mathbf{i}} \cap U_{\dot{\mathbf{j}}}$$

and

(2)
$$\phi_{ij} = \phi_j^{-1} | U_i \cap U_j \circ \phi_i | U_{ij}.$$

23.28 EXAMPLE Take
$$U_{ij} = \emptyset$$
 for all i,j -- then $X = \coprod_{i} X_{i}$.

[Note: If $\mathbf{A}_1,\dots,\mathbf{A}_n$ are nonzero commutative rings with unit, then

$$\underbrace{\prod_{i=1}^{n} \text{ Spec } A_{i} \approx \text{ Spec } (\prod_{i=1}^{n} A_{i})}_{i=1}$$

but for an infinite index set I, \coprod Spec A_i is not an affine scheme (it is not quasi-compact).]

23.29 LEMMA Let S be a scheme and let X $_i$ (i \in I), Y $_j$ (j \in J) be objects of SCH/S -- then

$$(\coprod_{i} X_{i}) \times_{S} (\coprod_{j} Y_{j}) \approx \coprod_{i,j} (X_{i} \times_{S} Y_{j}).$$

§24. MODULES

Let (X, O_X) be a ringed space.

24.1 DEFINITION An \mathcal{O}_{X} -module is a sheaf F of abelian groups on X such that \forall open subset $U \subset X$, the abelian group F(U) is a left $\mathcal{O}_{X}(U)$ -module and for each inclusion $V \subset U$ of open sets there is a commutative diagram

$$O_X(U) \times F(U) \longrightarrow F(U)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$O_X(V) \times F(V) \longrightarrow F(V).$$

24.2 NOTATION $\theta_{\rm X}$ -MOD is the category whose objects are the $\theta_{\rm X}$ -modules.

[Note: A morphism $F \to G$ of \mathcal{O}_X -modules is a morphism Ξ of sheaves of abelian groups such that \forall open subset $U \subset X$, the arrow $\Xi_U \colon F(U) \to G(U)$ is a homomorphism of left $\mathcal{O}_X(U)$ -modules. Denote the set of such by

$$\text{Hom}_{\mathcal{O}_{X}}(\mathcal{F},G)$$
.

Then this set is an abelian group which, moreover, is a left ΓO_X -module: Given $s \in \Gamma O_X$ and $E:F \to G$, define sE by the prescription

$$(s\Xi)_{U} = (s|U)\Xi_{U}$$

So, e.g., as left $\Gamma \mathcal{O}_X$ -modules,

$$\text{Hom}_{\mathcal{O}_{X}}(\mathcal{O}_{X}, F) \approx \Gamma F.$$

24.3 REMARK There is a standard list of operations that I shall not stop to

rehearse (kernel, cokernel, image, coimage,...).

- 24.4 EXAMPLE Let Z be the sheaf associated with the constant presheaf $U \rightarrow Z$ -then a Z-module is simply a sheaf of abelian groups on X.
 - 24.5 THEOREM $\theta_{_{\mathbf{Y}}}$ -MOD is an abelian category.
 - 24.6 THEOREM $\theta_{\rm X}$ -MOD has enough injectives.
 - 24.7 THEOREM $\theta_{_{
 m Y}}$ -MOD is complete and cocomplete.

[Any abelian category has equalizers and coequalizers.

ullet Given a set I and for each $i \in I$, an \mathcal{O}_{X} -module \mathcal{F}_{i} , the product

$$\prod_{i \in I} F_i$$

is the sheaf that assigns to each open subset $U \subset X$, the product

of left $\mathcal{O}_{\mathbf{X}}(\mathbf{U})$ -modules. It is also the categorical product.

ullet Given a set I and for each $\mathbf{i} \in \mathbf{I}$, an $\mathcal{O}_{\mathbf{X}}$ -module $\mathbf{f}_{\mathbf{i}}$, the direct sum

$$\bigoplus_{i \in T} F_i$$

is the sheaf associated with the presheaf that assigns to each open subset $U \subset X$, the direct sum

$$\underset{i \in I}{\oplus} F_i(U)$$

of left $\mathcal{O}_{\mathbf{X}}(\mathbf{U})$ -modules. It is also the categorical coproduct.]

24.8 DEFINITION Given θ_X -modules F and G, their tensor product

$$F \otimes_{\mathcal{O}_X} G$$

is the \mathcal{O}_X -module which is the sheaf associated with the presheaf that assigns to each open subset U \subset X, the tensor product

$$F(U) \otimes_{\mathcal{O}_{X}}(U) G(U)$$

of left $\mathcal{O}_{\mathbf{X}}(\mathbf{U})$ -modules.

24.9 DEFINITION Given θ_{X} -modules F and G, their <u>internal hom</u>

$$Hom_{\mathcal{O}_{\mathbf{X}}}$$
 (F,G)

is the \mathcal{O}_X -module which is the sheaf that assigns to each open subset U \subset X, the left \mathcal{O}_X (U)-module

$$\operatorname{Hom}_{\mathcal{O}_{\mathbf{X}}|\mathbf{U}}$$
 (F|U,G|U).

24.10 LEMMA Let F,G,H be \mathcal{O}_{X} -modules -- then

$$Hom_{\mathcal{O}_{\mathbf{X}}}(\mathsf{F} \otimes \mathcal{O}_{\mathbf{X}}^{G,H}) \approx Hom_{\mathcal{O}_{\mathbf{X}}}(\mathsf{F}, Hom_{\mathcal{O}_{\mathbf{X}}}^{G,H})$$
.

[Note: As left ΓO_{χ} -modules,

$$\operatorname{Hom}_{\mathcal{O}_{\mathbf{X}}}(F \otimes \mathcal{O}_{\mathbf{X}}^{G,H}) \approx \operatorname{Hom}_{\mathcal{O}_{\mathbf{X}}}(F,\operatorname{Hom}_{\mathcal{O}_{\mathbf{X}}}^{G,H}).$$

24.11 DEFINITION Suppose that

$$(X, \mathcal{O}_{X}) \xrightarrow{(f, f^{\sharp})} (Y, \mathcal{O}_{Y})$$

is a morphism of ringed spaces.

• Let F be an O_X -module. Form f_*F (an object of $\underline{Sh}(Y,\underline{AB})$) -- then f_*F

is an $f_* O_X$ -module, hence is an O_Y -module via the arrow $f^{\sharp} : O_Y \to f_* O_X$, call it res_f F.

• Let G be an \mathcal{O}_Y -module. Form f*G (an object of $\underline{\operatorname{Sh}}(X,\underline{\operatorname{AB}})$) -- then f*G is an $f*\mathcal{O}_Y$ -module. On the other hand, $f_\#:f*\mathcal{O}_Y\to\mathcal{O}_X$ is a morphism in $\underline{\operatorname{Sh}}(X,\underline{\operatorname{RNG}})$, thus

$$\mathcal{O}_{X} \boxtimes f * \mathcal{O}_{V} f * G$$

is an 0_{χ} -module, call it ext_f G.

24.12 EXAMPLE Take $G = O_{\mathbf{Y}}$ -- then

$$\text{ext}_{\text{f}} \mathcal{O}_{\text{Y}} \approx \mathcal{O}_{\text{X}}.$$

24.13 LEMMA The functor

$$\mathsf{ext}_{\mathtt{f}} \mathpunct{:} \mathcal{O}_{\mathtt{Y}} - \underline{\mathtt{MOD}} \longrightarrow \mathcal{O}_{\mathtt{X}} - \underline{\mathtt{MOD}}$$

is a left adjoint for the functor

$$\operatorname{res}_{\mathtt{f}}: \mathcal{O}_{\mathtt{X}} - \underline{\mathtt{MOD}} \longrightarrow \mathcal{O}_{\mathtt{Y}} - \underline{\mathtt{MOD}}.$$

24.14 REMARK Let

$$(X, \mathcal{O}_{X}) \xrightarrow{(f, f^{\sharp})} (Y, \mathcal{O}_{Y}) , (Y, \mathcal{O}_{Y}) \xrightarrow{(g, g^{\sharp})} (Z, \mathcal{O}_{Z})$$

be morphisms of ringed spaces — then the functors $\operatorname{res}_g \circ \operatorname{res}_f$ and $\operatorname{res}_g \circ f$ are equal while the functors $\operatorname{ext}_f \circ \operatorname{ext}_q$ and $\operatorname{ext}_q \circ f$ are naturally isomorphic.

24.15 NOTATION 0-MOD is the category whose objects are the triples (X, 0_{X} ,F),

where (X, \mathcal{O}_X) is a ringed space and F is an \mathcal{O}_X -module, and whose morphisms are the triples

$$(f,f^{\sharp},\Xi):(X,\mathcal{O}_{X},F) \rightarrow (Y,\mathcal{O}_{Y},G),$$

where $f:X \to Y$ is a continuous function, $f^{\#}:\mathcal{O}_{Y} \to f_{*}\mathcal{O}_{X}$ is a morphism in $\underline{Sh}(Y,\underline{RNG})$, $E:G \to f_{*}F$ is a morphism in $\underline{Sh}(Y,\underline{AB})$ such that \forall open subset $U \subset X$, the diagram

commutes.

24.16 LEMMA The projection

$$(X, \mathcal{O}_X, F) \rightarrow (X, \mathcal{O}_X)$$

is a fibration

$$P_{MOD}: O-\underline{MOD} \rightarrow \underline{TOP}_{RNG}.$$

PROOF Given (Y, O_Y, G) and

$$(x, O_x) \longrightarrow (y, O_y),$$

the composition

$$(x, \mathcal{O}_{X}, \mathcal{O}_{X} \boxtimes f * \mathcal{O}_{Y} f * G)$$

$$\longrightarrow (x, f * \mathcal{O}_{Y}, f * G) \longrightarrow (y, \mathcal{O}_{Y}, G)$$

is horizontal.

[Note: Recall that

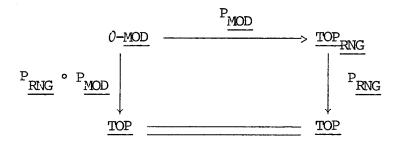
$$(id_{X}, f_{\sharp}) : (X, O_{X}) \rightarrow (X, f*O_{Y})$$

is a morphism of ringed spaces and there are arrows

$$\begin{array}{cccc}
 & O_{\mathbf{Y}} & \longrightarrow & f_{\star}f^{\star}O_{\mathbf{Y}} \\
 & G & \longrightarrow & f_{\star}f^{\star}G
\end{array}$$

of adjunction.]

24.17 REMARK The commutative diagram



is thus an instance of 6.2.

§25. QUASI-COHERENT MODULES

Let (X, O_X) be a ringed space.

25.1 NOTATION Given a set I and an O_X -module F, write $F^{(I)}$ for the direct sum $\bigoplus_{i \in I} F_i \quad (\forall i, F_i = F).$

25.2 DEFINITION An \mathcal{O}_X -module F is said to be <u>quasi-coherent</u> if \forall $x \in X$, there exists an open neighborhood U of x, sets I and J (depending on x), and an exact sequence

$$(\mathcal{O}_{\mathbf{X}}|\mathbf{U}) \xrightarrow{(\mathbf{I})} \longrightarrow (\mathcal{O}_{\mathbf{X}}|\mathbf{U}) \xrightarrow{(\mathbf{J})} \longrightarrow F|\mathbf{U} \longrightarrow 0$$

of O_X | U-modules.

- 25.3 NOTATION QCO(X) is the full subcategory of θ_X -MOD whose objects are the quasi-coherent θ_X -modules.
 - 25.4 REMARK In general, QCO(X) is not an abelian category.
 - 25.5 LFMMA Let F,G be quasi-coherent \mathcal{O}_X -modules -- then $F \oplus G$ is quasi-coherent.

[Note: An infinite direct sum of quasi-coherent \mathcal{O}_X -modules need not be quasi-coherent.]

25.6 LEMMA Let F,G be quasi-coherent \mathcal{O}_X -modules -- then $F \otimes_{\mathcal{O}_X} G$ is quasi-coherent.

[Note: On the other hand, $\operatorname{Hom}_{\mathcal{O}_X}$ (F,G) need not be quasi-coherent.]

N.B. QCO(X) is a symmetric monoidal category under the tensor product (the unit is $\theta_{\rm X}$).

25.7 DEFINITION An \mathcal{O}_X -module F is said to be <u>locally free</u> if \forall $x \in X$, there exists an open neighborhood U of x and a set I (depending on x) such that F|U is isomorphic to $(\mathcal{O}_X|U)^{(I)}$ as an $\mathcal{O}_X|U$ -module.

25.8 LEMMA A locally free $\theta_{\rm X}$ -module F is necessarily quasi-coherent.

25.9 LEMMA Suppose that

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

is a morphism of ringed spaces.

- \bullet Let F be a quasi-coherent \mathcal{O}_X -module -- then res_f F is not necessarily a quasi-coherent \mathcal{O}_V -module.
- \bullet Let G be a quasi-coherent $\mathcal{O}_Y\text{-module}$ then ext_{f} G is necessarily a quasi-coherent $\mathcal{O}_X\text{-module}.$

25.10 CONSTRUCTION Let (X, \mathcal{O}_X) be a ringed space. Suppose that $A \in Ob$ RNG and $\phi: A \to \Gamma\mathcal{O}_X$ (= $\mathcal{O}_X(X)$) is a ring homomorphism. Let M be a left A-module. Consider the canonical arrow

$$(\pi,\pi^{\sharp}):(X,\mathcal{O}_{X})\longrightarrow (\star,\mathcal{O}_{\star}),$$

where $\mathcal{O}_{\star}\star$ = A (π^{\sharp} = φ) — then ext $_{\pi}$ M is quasi-coherent. In addition, the assignment M \rightarrow ext $_{\pi}$ M

defines a functor

$$A-MOD \rightarrow QCO(X)$$

and given any $\theta_{\mathbf{x}}$ -module F,

$$\text{Hom}_{\mathcal{O}_X}(\text{ext}_{\pi} M, F) \approx \text{Hom}_{A}(M, \Gamma F),$$

where the left A-module structure on FF comes from the left F0_X-module structure via φ .

25.11 REMARK One can take A = ΓO_{X} , ϕ = id, in which case it is customary to write F_{M} in place of ext $_{\pi}$ M.

Given $A \in Ob \ \underline{RNG}$, we shall now recall the connection between $A-\underline{MOD}$ and $\underline{QCO}(\operatorname{Spec}\ A)$. So in 25.10, take $(X,\mathcal{O}_{X})=(\operatorname{Spec}\ A,\mathcal{O}_{A})$ (hence $\Gamma\mathcal{O}_{A}\approx A)$ — then for every left A-module M, the sheaf \widetilde{M} is canonically isomorphic to F_{M} (and this isomorphism is functorial in M). Therefore the \widetilde{M} are quasi-coherent and given any \mathcal{O}_{X} -module F,

$$\operatorname{Hom}_{\mathcal{O}_{\mathbf{A}}}(\widetilde{\mathbf{M}},\mathcal{F}) \approx \operatorname{Hom}_{\mathbf{A}}(\mathbf{M},\Gamma\mathcal{F}).$$

25.12 LEMMA For all left A-modules M and N,

$$\text{Hom}_{\mathcal{O}_{\widetilde{A}}}(\widetilde{M},\widetilde{N}) \approx \text{Hom}_{\widetilde{A}}(M,N)$$
.

[Bear in mind that

25.13 LEMMA For every quasi-coherent $\mathcal{O}_{\mathbf{A}}$ -module $\mathcal{F}_{\mathbf{A}}$

$$(\Gamma F)^{\sim} \approx F.$$

25.14 THEOREM The functor

$$\sim$$
:A-MOD \rightarrow QCO(Spec A)

that sends M to $\widetilde{\mathrm{M}}$ is an equivalence of categories.

[In fact, ~ is fully faithful (cf. 25.12) and has a representative image (cf. 25.13).]

- 25.15 EXAMPLE The category of abelian groups is equivalent to $QCO(Spec \ \underline{Z})$.
- 25.16 LEMMA Let A,B ∈ Ob RNG, suppose that

$$(f,f^{\sharp}):(Spec\ B,\mathcal{O}_{B}) \rightarrow (Spec\ A,\mathcal{O}_{A})$$

is a morphism of affine schemes, and let $\rho:A\to B$ be the associated ring homomorphism.

• For every left B-module N,

$$res_f \tilde{N} \approx (res_o N)^{\sim}$$

functorially in N.

• For every left A-module M,

$$\operatorname{ext}_{f} \tilde{M} \approx (\operatorname{ext}_{\rho} \tilde{M})^{\sim}$$

functorially in M.

25.17 REMARK There is a functor

$$(Spec, 0, \sim) : MOD(AB)^{OP} \longrightarrow 0-MOD$$

which sends an object (A,M) to

(Spec A,
$$\mathcal{O}_{\Delta}$$
, \widetilde{M})

and which sends a morphism $(f,\phi):(A,M) \to (B,N)$ to

$$(\operatorname{Spec} \ \operatorname{f}, \mathcal{O}_{\operatorname{f}}, \overset{\sim}{\phi}) : (\operatorname{Spec} \ \operatorname{B}, \mathcal{O}_{\operatorname{B}}, \overset{\sim}{\operatorname{N}}) \longrightarrow (\operatorname{Spec} \ \operatorname{A}, \mathcal{O}_{\operatorname{A}}, \overset{\sim}{\operatorname{M}}) \ .$$

[Note: On a principal open set D(a) ($a \in A$), $\tilde{M}(D(a)) = \tilde{M}$ and

$$((Spec f)_* \tilde{N})(D(a)) = \tilde{N}(D(f(a))) = N_{f(a)}.$$

Furthermore, there are arrows of localization

$$\begin{bmatrix} A & \longrightarrow & A_{a} & & & M & \longrightarrow & M_{a} \\ B & \longrightarrow & B_{f(a)} & & & & N & \longrightarrow & N_{f(a)} \end{bmatrix}$$

and a commutative diagram

$$(A,M) \longrightarrow (A_a,M_a)$$

$$\downarrow \qquad \qquad \downarrow$$

$$(B,N) \longrightarrow (B_{f(a)},N_{f(a)}) .$$

It remains to consider the pairs (X, \mathcal{O}_{X}), where X is a scheme.

[Note: It has been shown by Rosenberg † that (X, \mathcal{O}_X) can be reconstructed up to isomorphism from QCO(X).]

25.18 LEMMA Let F be an \mathcal{O}_{X} -module -- then F is quasi-coherent iff for every

 $^{^{\}dagger}$ Lecture Notes in Pure and Applied Mathematics $\underline{197}$ (1998), 257-274.

affine open U \subset X (U \approx Spec A), the restriction F|U is of the form M for some M in A-MOD.

N.B. If F is a quasi-coherent \mathcal{O}_X -module, then for all affine open U, V with $V \subset U$, the canonical arrow

$$\mathcal{O}_{X}(V) \otimes_{\mathcal{O}_{X}(U)} F(U) \rightarrow F(V)$$

is an isomorphism of $\mathcal{O}_{_{\mathbf{X}}}(\mathbf{V})$ -modules.

25.19 LEMMA Suppose that

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

is a morphism of schemes. Let G be a quasi-coherent \mathcal{O}_Y -module -- then $\operatorname{ext}_f G$ is a quasi-coherent \mathcal{O}_X -module (cf. 25.9).

25.20 REMARK The notation used in 7.3 is suggestive but misleading: Replace f* by ext_f

25.21 LEMMA Suppose that

$$(X, O_X) \xrightarrow{(f, f^{\sharp})} (Y, O_Y)$$

is a morphism of schemes, where f is quasi-compact and quasi-separated. Let F be a quasi-coherent 0_X -module -- then res_f F is a quasi-coherent 0_Y -module (cf. 25.9).

25.22 REMARK If U is an open subset of a scheme X, then in general, res ${\bf i}_{\rm U}$ $^0{\rm X}^{|\rm U}$ is not quasi-coherent.

25.23 THEOREM QCO(X) is an abelian category.

25.24 RAPPEL A <u>Grothendieck category</u> is a cocomplete abelian category in which filtered colimits commute with finite limits or, equivalently, in which filtered colimits of exact sequences are exact.

N.B. In a Grothendieck category, every filtered colimit of monomorphisms is a monomorphism, coproducts of monomorphisms are monomorphisms, and

$$t: \coprod_{i} X_{i} \to \prod_{i} X_{i}$$

is a monomorphism.

25.25 EXAMPLE Let A be a commutative ring with unit — then A-MOD is Grothen-dieck.

[Note: In particular, AB is Grothendieck but its full subcategory whose objects are the finitely generated abelian groups is not Grothendieck.]

25.26 THEOREM QCO(X) is a Grothendieck category.

25.27 DEFINITION Given a locally small category \underline{C} , an object \underline{U} in \underline{C} is said to be a <u>separator</u> for \underline{C} if the functor $Mor(\underline{U}, \longrightarrow) : \underline{C} \to \underline{SET}$ is faithful, i.e., if for every pair $f,g:X \to Y$ of distinct morphisms, there exists a morphism $\sigma:U \to X$ such that $f \circ \sigma \neq g \circ \sigma$.

25.28 EXAMPLE Let A be a commutative ring with unit -- then A, viewed as a left A-module, is a separator for A-MOD.

25.29 THEOREM QCO(X) admits a separator.

N.B. Every Grothendieck category with a separator is complete and has enough injectives.

25.30 REMARK It can be shown that $\underline{QCO}(X)$ is a coreflective subcategory of \mathcal{O}_{V} -MOD, i.e., the inclusion functor

$$\overline{QCO}(X) \rightarrow \mathcal{O}_X - \overline{MOD}$$

has a right adjoint.

Fix a regular cardinal K.

- 25.31 DEFINITION Let C be a locally small cocomplete category then an object $X \in Ob$ C is κ -definite if Mor(X,—) preserves κ -filtered colimits.
 - 25.32 EXAMPLE In TOP, no nondiscrete X is κ -definite.
- 25.33 DEFINITION Let \underline{C} be a locally small cocomplete category then \underline{C} is $\underline{\kappa}$ -presentable if up to isomorphism, there exists a set of κ -definite objects and every object in C is a κ -filtered colimit of κ -definite objects.
- 25.34 EXAMPLE <u>SET</u> and <u>CAT</u> are \aleph_0 -presentable but <u>TOP</u> is not κ -presentable for any κ .
- 25.35 DEFINITION Let \underline{C} be a locally small cocomplete category -- then \underline{C} is presentable if C is κ -presentable for some κ .

[Note: Every presentable category is cocomplete (by definition) and complete, wellpowered and cowellpowered.]

25.36 THEOREM (Beke †) Suppose that $\underline{\mathbf{C}}$ is a Grothendieck category with a

[†] Math. Proc. Camb. Phil. Soc. 129 (2000), 447-475.

separator -- then $\underline{\mathbf{C}}$ is presentable.

25.37 APPLICATION $\underline{QCO}(X)$ is presentable.

§26. LOCAL TRIVIALITY

Let C be a category.

- 26.1 DEFINITION A subcategory of <u>trivial objects</u> is a replete subcategory of C.
- 26.2 EXAMPLE If \underline{C} has initial objects, then the associated full subcategory is isomorphism closed, hence is a subcategory of trivial objects.
- 26.3 EXAMPLE If \underline{C} has final objects, then the asociated full subcategory is isomorphism closed, hence is a subcategory of trivial objects.

Let \underline{A} be a category, $F:\underline{A} \rightarrow \underline{C}$ a functor.

- 26.4 DEFINITION The replete full image of F is the isomorphism closed full subcategory of \underline{C} whose objects are those objects which are isomorphic to some FA $(A \in Ob \ \underline{A})$.
- 26.5 EXAMPLE Take $\underline{A} = \underline{SET}$, $\underline{C} = \underline{GR}$, $F:\underline{A} \to \underline{C}$ the left adjoint to the forgetful functor then the replete full image of F is the category of free groups.
- 26.6 EXAMPLE Take $\underline{A} = \underline{RNG}^{OP}$, $\underline{C} = \underline{LOC-TOP}_{\underline{RNG}}$, $F:\underline{A} \to \underline{C}$ the functor that sends A to (Spec $A, \mathcal{O}_{\underline{A}}$) then the replete full image of F is the category of affine schemes.

Let $\underline{T} \subset \underline{C}$ be a subcategory of trivial objects.

26.7 DEFINITION Let C be a covering of an object X in C — then X is <u>locally trivial</u> (w.r.t. T) if the domain of each $g \in C$ is in T.

26.8 DEFINITION Let κ be a covering function on \underline{C} — then an object X in \underline{C} is locally trivial (w.r.t. \underline{T}) if it is locally trivial (w.r.t. \underline{T}) for some $C \in \kappa_{\underline{X}}$.

N.B. To ensure that

"trivial" => "locally trivial",

it suffices to assume that $\forall~T\in Ob~\underline{\mathtt{T}}\text{, }\{id_{\overline{\mathtt{T}}}\text{:}\mathtt{T}~\rightarrow~\mathtt{T}\}\in \kappa_{\overline{\mathtt{T}}}\text{.}$

26.9 REMARK Suppose that \forall X \in Ob C, $\kappa_{X} = \{id_{X}: X \rightarrow X\}$ -- then for any \underline{T} , the locally trivial objects are the trivial objects.

26.10 EXAMPLE Take C = SET.

- Let \underline{T} be the subcategory whose only object is the empty set \emptyset and whose only morphism is $\mathrm{id}_{\emptyset} : \emptyset \to \emptyset$. Define a covering function κ by setting $\kappa_{\mathrm{X}} = \{\emptyset \to \mathrm{X}\}$ -- then all objects are locally trivial.
- Let \underline{T} be the subcategory whose objects are the singletons. Define a covering function κ by setting $\kappa_{\emptyset}=\mathrm{id}_{\emptyset}$ and

$$\kappa_{X} = \{\{x\} \rightarrow X : x \in X\} (X \neq \emptyset).$$

Then all objects are locally trivial.

26.11 EXAMPLE Take $\underline{C} = \underline{TOP}$, let κ be the open subset coverage (cf. 11.20), and take for \underline{T} the euclidean spaces, i.e., the topological spaces which are homeomorphic to some open subset of some R^n — then the locally trivial objects are the topological manifolds.

[Note: To say that X is a topological manifold means that X admits a covering

by open sets $U_i \subset X$, where \forall i, U_i is homeomorphic to an open subset of R^n in depends on i).]

26.12 EXAMPLE Take $\underline{C} = \underline{LOC-TOP}_{\underline{RNG}}$, let κ be the open subset coverage, and take for \underline{T} the affine schemes -- then the locally trivial objects are the schemes (cf. 23.19).

[Note: An open subset U of a locally ringed space (X, \mathcal{O}_X) can be viewed as a locally ringed space (let $\mathcal{O}_U = \mathcal{O}_X | U$), thus it makes sense to consider the open subset coverage.]

26.13 EXAMPLE Take $\underline{C} = \underline{TOP}_{\underline{RNG}}$, let κ be the open subset coverage, and take $\underline{T} = \underline{LOC-TOP}_{\underline{RNG}}$ (which is replete (cf. 23.11)) — then here, all locally trivial objects are trivial.

[Note: If U < X is open, then the stalk of \mathcal{O}_U at an $x \in U$ is \mathcal{O}_{X,x^*}]

Consider a one point ringed space $(\{x\}, 0_{\{x\}})$ — then $0_{\{x\}}\emptyset = \{0\}$ (a zero ring), $0_{\{x\}}\{x\} = A$ (a ring). Abbreviate this setup to $(\{x\}, A)$ — then a morphism

$$(\{x\},A) \xrightarrow{(f,f^{\sharp})} (\{y\},B)$$

of ringed spaces is simply a homomorphism $f^{\sharp}:B\to A$.

26.14 EXAMPLE Let \underline{T} be the replete subcategory of $\underline{TOP}_{\underline{RNG}}$ whose objects are the pairs ($\{x\}$,A), where A is a local ring, and whose morphisms are the morphisms

$$(\{x\},A) \xrightarrow{(f,f^{\sharp})} (\{y\},B)$$

of ringed spaces such that the momomorphism $f^{\sharp}:B \to A$ is a local homomorphism.

Define a covering function κ on $\underline{\text{TOP}}_{\underline{\text{RNG}}}$ by setting $\kappa_{(\emptyset, \mathcal{O}_{\emptyset})} = \mathrm{id}_{(\emptyset, \mathcal{O}_{\emptyset})}$ and

$$\kappa_{(\mathbf{X},\mathcal{O}_{\mathbf{X}})} = \{(\{\mathbf{x}\},\mathcal{O}_{\mathbf{X},\mathbf{x}}) \rightarrow (\mathbf{X},\mathcal{O}_{\mathbf{X}}) : \mathbf{x} \in \mathbf{X}\} \ (\mathbf{X} \neq \emptyset).$$

Then the locally trivial objects are the locally ringed spaces.

Let $P: \underline{E} \to \underline{B}$ be a fibration. Suppose that $\underline{T} \subset \underline{E}$ is a subcategory of trivial objects and let κ be a covering function on \underline{B} .

26.15 DEFINITION An object $X \in Ob \ \underline{E}$ is <u>locally trivial</u> (w.r.t. \underline{T}) if it is locally trivial (w.r.t. \underline{T}) for some $C \in (P^{-1}\kappa)_X$.

[Note: This reduces to 26.8 if E = B, P = id.]

Let $P: \underline{E} \to \underline{B}$ be a fibration. Suppose that \underline{B} has a final object $\star_{\underline{B}}$ and that $\underline{E}_{\star} \neq 0$. Let \underline{C} be a subcategory of \underline{E}_{\star} . Denote by $\underline{T}_{\underline{C}}$ the full subcategory of \underline{E} whose objects are the X for which there exists an object $C \in C$ ob \underline{C} and a horizontal arrow $X \to C$.

- 26.16 LEMMA $\underline{\mathbf{T}}_{\underline{\mathbf{C}}}$ is a replete subcategory of $\underline{\mathbf{E}}$.
- 26.17 REMARK There is an analogous statement involving opfibrations with trivial objects determined by a subcategory of the fiber over an initial object.
- 26.18 EXAMPLE Consider the fibration $P_{\underline{A}}:\underline{TOP_{\underline{A}}}\to \underline{TOP}$ of 22.11. Place on \underline{TOP} the open subset coverage κ and take for \underline{C} the fiber over a singleton *, thus the objects of $\underline{T}_{\underline{C}}$ are the \underline{A} -spaces $(X, \mathcal{O}_{\underline{X}})$ which are the domain of a horizontal arrow $(X, \mathcal{O}_{\underline{X}}) \to (*, \mathcal{O}_{*})$ over $!: X \to *$ for some \mathcal{O}_{*} .

• The trivial objects are the (X,0_X) such that $0_X \approx !*0_*$ ($\approx \Delta \circ \text{ev } 0_*$ (cf. 22.19)).

[The point is that for any X, the arrow

$$(!, \mu_{0_*}): (X, !*0_*) \rightarrow (*, 0_*)$$

is horizontal (cf. 22.11).]

Observe next that if U is an open subset of X, then

$$i_{II}^*:\underline{Sh}(X,\underline{A}) \rightarrow \underline{Sh}(U,\underline{A})$$

and $\forall \ \mathcal{O}_{\mathbf{X'}}$ the arrow

$$(\mathtt{i}_{\mathtt{U}},\mu_{\mathcal{O}_{\mathtt{X}}}):(\mathtt{U},\mathcal{O}_{\mathtt{X}}|\mathtt{U})\to(\mathtt{X},\mathcal{O}_{\mathtt{X}})$$

is horizontal (cf. 22.12). So, if $X = \bigcup_{i \in I} U_i$, then

$$\{(\mathbf{i}_{\mathbf{U}_{\mathbf{i}}}, \mu_{\mathcal{O}_{\mathbf{X}}}) : (\mathbf{U}_{\mathbf{i}}, \mathcal{O}_{\mathbf{X}} | \mathbf{U}_{\mathbf{i}}) \rightarrow (\mathbf{X}, \mathcal{O}_{\mathbf{X}})\} \in (\mathbf{P}_{\underline{\mathbf{A}}}^{-1} \kappa)_{(\mathbf{X}, \mathcal{O}_{\mathbf{X}})}.$$

• The locally trivial objects are the (X, 0_X) such that X admits an open covering {U_i:i \in I} with the following property: \forall i,

$$O_{X}|U_{i} \approx !_{i}^{*}(O_{*})_{i}$$
.

[Note: $!_{i}^{*}$ is calculated per U_{i} , hence

$$!_{i}^{*}:\underline{Sh}(*,\underline{A}) \rightarrow \underline{Sh}(U_{i},\underline{A})$$

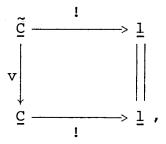
and $(0_*)_i$ is an object in $\underline{\operatorname{Sh}}(*,\underline{\operatorname{A}})$ that depends on i.]

26.19 EXAMPLE Consider the fibration Ob: $\underline{CAT} \to \underline{SET}$ of 5.1. Place on \underline{SET} the "inclusion of elements" coverage κ (cf. 26.10) and take for \underline{C} the singleton $\{\underline{1}\}$ in the fiber over *, thus the objects of $\underline{T}_{\underline{C}}$ are the small categories \underline{C} such that

 $\underline{\underline{C}} \longrightarrow \underline{\underline{l}}$ is horizontal.

• The trivial objects $\neq \underline{0}$ are the small categories \underline{C} such that \forall X,Y \in Ob \underline{C} , #Mor(X,Y) = 1.

[Assume first that \underline{C} is trivial and pass to the arrow Ob $\underline{C} \to *$. Proceeding as in 5.1, construct a category $\underline{\widetilde{C}}$ and a horizontal $\underline{\widetilde{C}} \to \underline{1}$ such that Ob! is $\underline{!}$ Ob $\underline{C} \longrightarrow *$ -- then $\forall \ \widetilde{X}, \widetilde{Y} \in \mathsf{Ob} \ \underline{\widetilde{C}}, \ \#\mathsf{Mor}(\widetilde{X}, \widetilde{Y}) = 1.$ But since $\underline{C} \longrightarrow \mathsf{1}$ is horizontal, there is a vertical isomorphism $v: \underline{\widetilde{C}} \to \underline{C}$ and a commutative diagram



so \forall X,Y \in Ob C, #Mor(X,Y) = 1, which settles the necessity. Turning to the sufficiency, consider a setup

the claim being that there exists a unique functor $v: \underline{C}_0 \to \underline{C}$ such that Ob v = x and ! $\circ v = w$. This, however, is obvious: Define v on an object X_0 by $vX_0 = xX_0$ and on a morphism $f_0: X_0 \to Y_0$ by $vf_0 = f$, the unique element of $Mor(xX_0, xY_0)$.

N.B. The arrow $\underline{0} \longrightarrow \underline{1}$ is horizontal. Therefore $\underline{0}$ is trivial.

[In the foregoing, let $\underline{C} = \underline{0}$ — then Ob $\underline{C} = \emptyset$, hence Ob $\underline{C}_0 = \emptyset$ and $\underline{x} = \mathrm{id}_{\emptyset}$. And this means that $\underline{C}_0 = \underline{0}$, so $\underline{v} = \mathrm{id}_{\underline{0}}$.]

By definition, if $C \neq 0$, then

$$\kappa_{\mbox{Ob }\underline{C}} = \{\{x\} \longrightarrow \mbox{Ob }\underline{C} \colon x \in \mbox{Ob }\underline{C}\}.$$

Choose a horizontal $u_X: \underline{C}_X \to \underline{C}$ such that Ob $u_X = i_X$, thus Ob $\underline{C}_X = \{x\}$. And

$$\{\underline{c}_{X} \xrightarrow{u_{X}} \underline{c} : X \in Ob \underline{c}\} \in (Ob^{-1}\kappa)_{\underline{c}}.$$

• The locally trivial objects $\neq \underline{0}$ are the small categories \underline{C} such that $\forall \ X \in Ob \ \underline{C}$, $Mor(X,X) = \{id_X\}$.

[Construct \underline{C}_X as in 5.1, thus \forall X \in Ob \underline{C} ,

$$Mor_{\underline{C}_{X}}(X,X) = \{X\} \times Mor(X,X) \times \{X\},$$

implying thereby that

$$\#Mor_{\underline{C}_{X}}(X,X) = 1 \iff \#Mor(X,X) = 1.$$

E.g.: Every set viewed as a discrete category is locally trivial.

26.20 EXAMPLE Viewing R as a topological ring, given a topological space B, let

$$\theta_{R} = (B \times R \rightarrow B)$$
.

Then θ_B is an internal ring in <u>TOP/B</u>. This said, denote by \underline{M}_B the category whose objects are the internal θ_B -modules.

(*) Take B = * -- then \underline{M}_B is the category of real topological vector spaces. Define a pseudo functor $F:\underline{TOP}^{OP} \to 2\text{-CAT}$ by sending B to \underline{M}_B and $\beta:B \to B'$ to $F\beta:\underline{M}_{B'} \to \underline{M}_B$ ("pullback"). Use now the notation of 7.7 and form $gro_{\underline{TOP}}F$, the objects of which are the pairs (B,M), where $B \in Ob$ \underline{TOP} and $M \in Ob$ FB, and whose morphisms

are the arrows $(\beta,f):(B,M)\to (B',M')$, where $\beta\in Mor(B,B')$ and $f\in Mor(M,(F\beta)M')$.

Consider the fibration $\Theta_F: gro_{\underline{TOP}}F \to \underline{TOP}$ of 7.9. Place on \underline{TOP} the open subset coverage κ and take for \underline{C} the subcategory of the fiber over * whose objects are the R^n , thus the objects of $\underline{T}_{\underline{C}}$ are the pairs (B,M) which are the domain of a horizontal arrow (B,M) \to (*, R^n) over !:B \to * for some R^n .

• The trivial objects are the (B,M) such that $M \approx B \times R^n$. [The point is that for any B, the morphism

$$(!, id_{(F!)R^n}) : (B, (F!)R^n) \rightarrow (*, R^n)$$

is horizontal (cf. 7.12) and (F!) $R^n \approx B \times R^n$.]

Observe next that if U is an open subset of B, then $Fi_U:\underline{M}_B\to \underline{M}_U$. Agreeing to write M|U in place of $(Fi_U)M$, the arrow

$$(i_U, id_{M|U}) : (U, M|U) \rightarrow (B, M)$$

is horizontal (cf. 7.12). So if B = $\bigcup_{i \in I} U_i$, then

$$\{(i_{U_{i}}, id_{M|U_{i}}): (U_{i}, M|U_{i}) \rightarrow (B, M)\} \in (\Theta_{F}^{-1} \ltimes)_{(B, M)}.$$

• The locally trivial objects are the (B,M) such that B admits an open covering $\{U_i:i\in I\}$ with the following property: \forall i,

$$M|U_{i} \approx U_{i} \times R^{n_{i}}$$

[Note: Here n depends on i and the isomorphism is computed in $\underline{\textbf{M}}_{\textbf{i}}$.]

26.21 RAPPEL The triple $\langle \underline{AB}, \emptyset, \overline{I} \rangle$ is a symmetric monoidal category and the

commutative monoids therein are the commutative rings with unit.

26.22 NOTATION Given A ∈ Ob RNG, let A-MOD be the category of left A-modules.

Let A,B be commutative rings with unit and suppose that $f:A \to B$ is a ring homomorphism — then there is a functor

and a functor

26.23 LEMMA The functor $\operatorname{ext}_{\mathbf{f}}$ is a left adjoint for the functor $\operatorname{res}_{\mathbf{f}}$.

26.24 NOTATION MOD(AB) is the category whose objects are the pairs (A,M), where A is a commutative ring with unit and M is a left A-module, and whose morphisms are the arrows $(f,\phi):(A,M) \to (B,N)$, where $f:A \to B$ is a ring homomorphism and $\phi:M \to N$ is a morphism in AB such that the diagram

commutes, the vertical arrows being the actions of A and B on M and N.

26.25 REMARK There is a 2-functor

that sends A to A-MOD and $f:R \to S$ to $res_{f}:B-MOD \to A-MOD$. Its Grothendieck

construction $\text{gro}_{\overline{RNG}}F$ can be identified with $\underline{\text{MOD}}(\overline{AB})$.

[Note: There is a pseudo functor

that sends A to A-MOD and f:A \rightarrow B to ext_f:A-MOD \rightarrow B-MOD.]

26.26 LEMMA The projection $(A,M) \rightarrow A$ defines a fibration

$$P_{AB}: \underline{MOD}(\underline{AB}) \rightarrow \underline{RNG}.$$

PROOF Given (B,N) and f:A → B, the morphism

$$(A, res_f N) \rightarrow (B, N)$$

is horizontal.

26.27 LEMMA The projection (R,M) → R defines an opfibration

$$P_{AB}: \underline{MOD}(\underline{AB}) \rightarrow \underline{RNG}.$$

PROOF Given (A,M) and $f:A \rightarrow B$, the morphism

$$(A,M) \rightarrow (B,B \otimes_A M)$$

is ophorizontal.

26.28 REMARK Therefore $\boldsymbol{P}_{\! A\! B}$ is a bifibration (cf. 5.15).

26.29 EXAMPLE Consider the optibration $P_{\underline{AB}}:\underline{MOD}(\underline{AB})\to \underline{RNG}$ of 26.27. Place on \underline{RNG} the Zariski coverage κ (cf. 11.16) and bearing in mind 26.17, take for \underline{C} the subcategory of the fiber over Z whose objects are the Z^n , thus the objects of $\underline{T}_{\underline{C}}$ are the pairs (A,M) which are the codomain of an ophorizontal arrow $(Z,Z^n)\to (A,M)$ over $Y:Z\to A$ for some Z^n .

- The trivial objects are the (A,M) such that M is a free left A-module of finite rank.
- The locally trivial objects are the (A,M) such that M is a finitely generated projective left A-module.

APPENDIX

Fix a topological group G and consider the fibration G-BUN(TOP) \rightarrow TOP of 5.3 — then its fiber G-BUN(TOP) $_*$ over * is (isomorphic to) $\underline{\text{MOD}}_{G}$, the category of right G-modules over the monoid G in $\underline{\text{TOP}}$. Take for $\underline{\text{C}}$ the singleton subcategory $\{G \rightarrow *\}$, thus the objects of $\underline{\text{T}}_{\underline{\text{C}}}$ are the X \rightarrow B which are isomorphic to a product X \times G \rightarrow B.

- Place on <u>TOP</u> the open subset coverage then the locally trivial objects over B are those objects $X \to B$ in $\underline{PRIN}_{B,G}$ for which there exists an open covering $\{U_{\underline{i}}: i \in I\}$ of B such that $\forall i$, $X|U_{\underline{i}} \approx U_{\underline{i}} \times G$ in $\underline{PRIN}_{U_{\underline{i}},G}$.
- Place on \underline{TOP} the open map coverage (cf. 11.19) then the locally trivial objects over B are the objects $X \to B$ of $\underline{PRIN}_{B,G}$.

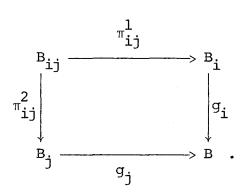
STACKS

Let \underline{B} be a category equipped with a Grothendieck coverage κ such that $\forall \ B \in Ob \ \underline{B}\text{, } \{id_{\underline{B}}\text{:}B \to B\} \in \kappa_{\underline{B}}\text{.}$

 $\underline{\mathtt{ST-1}} \colon \ \, \mathtt{NOTATION} \,\, \mathtt{Given} \,\, \{\mathtt{g}_{\mathbf{i}} \colon \mathtt{B}_{\mathbf{i}} \, \to \, \mathtt{B}\} \, \in \, \kappa_{\mathtt{B}}' \,\, \mathtt{put}$

$$B_{ij} = B_i \times_B B_j$$

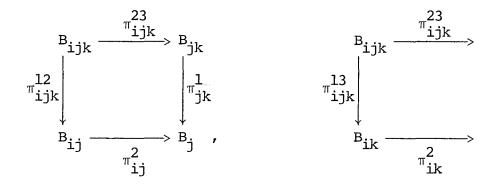
and define π_{ij}^1 , π_{ij}^2 per the pullback square



 $\underline{\mathtt{ST-2}} \colon \ \, \mathtt{NOTATION} \,\, \mathtt{Given} \,\, \{\mathtt{g}_{\mathbf{i}} \colon \mathtt{B}_{\mathbf{i}} \, \to \, \mathtt{B} \} \, \in \, \kappa_{\mathtt{B}}^{}, \,\, \mathtt{put}$

$$B_{ijk} = B_i \times_B B_j \times_B B_k$$

and define $\pi^{12}_{ijk'}$, $\pi^{13}_{ijk'}$, π^{23}_{ijk} by the pullback squares



Let $F:\underline{B}^{\mathrm{OP}} \to 2\text{-CAT}$ be a pseudo functor (cf. §3).

ST-3: DEFINITION A set of descent data on $\{g_i:B_i \to B\} \in \kappa_B$ is a collection of objects $X_i \in FB_i$ and a collection of isomorphisms

$$\phi_{ij}: F(\pi_{ij}^2) X_j \rightarrow F(\pi_{ij}^1) X_i$$

in FB; which satisfy the cocycle condition

$$F(\pi_{\mathbf{i}\mathbf{j}\mathbf{k}}^{13})\phi_{\mathbf{i}\mathbf{k}} = F(\pi_{\mathbf{i}\mathbf{j}\mathbf{k}}^{12})\phi_{\mathbf{i}\mathbf{j}} \circ F(\pi_{\mathbf{i}\mathbf{j}\mathbf{k}}^{23})\phi_{\mathbf{j}\mathbf{k}}$$

in ${\rm FB}_{\mbox{\scriptsize ijk}}$ modulo the "coherency" implicit in F.

[Spelled out, the demand is that the composition

$$F(\pi_{\mathbf{i}jk}^{23})F(\pi_{\mathbf{j}k}^{2})X_{k}$$

$$\xrightarrow{\gamma_{\mathbf{i}jk}^{23}, \pi_{\mathbf{j}k}^{2}} \longrightarrow F(\pi_{\mathbf{j}k}^{2} \circ \pi_{\mathbf{i}jk}^{23})X_{k}$$

$$\xrightarrow{-1} F(\pi_{\mathbf{i}k}^{2} \circ \pi_{\mathbf{i}jk}^{13})X_{k}$$

$$\xrightarrow{\gamma_{\mathbf{i}jk}^{-1}, \pi_{\mathbf{i}k}^{2}} \longrightarrow F(\pi_{\mathbf{i}jk}^{13}) \circ F(\pi_{\mathbf{i}k}^{2})X_{k}$$

$$\xrightarrow{F(\pi_{\mathbf{i}jk}^{13})\phi_{\mathbf{i}k}} \longrightarrow F(\pi_{\mathbf{i}jk}^{13}) \circ F(\pi_{\mathbf{i}k}^{1})X_{\mathbf{i}}$$

is the same as the composition

$$F(\pi_{ijk}^{23})F(\pi_{jk}^{2})X_{k}$$

$$\frac{F(\pi_{ijk}^{23}) \phi_{jk}}{\longrightarrow} F(\pi_{ijk}^{23}) F(\pi_{jk}^{1}) X_{j}$$

$$\frac{Y_{\pi_{ijk}^{23}, \pi_{jk}^{1}}^{1} X_{j}}{\longrightarrow} F(\pi_{jk}^{1} \circ \pi_{ijk}^{23}) X_{j}$$

$$\frac{Y_{\pi_{ijk}^{12}, \pi_{jk}^{2}}^{1} X_{j}}{\longrightarrow} F(\pi_{ijk}^{12}) \circ F(\pi_{ij}^{2}) X_{j}$$

$$\frac{F(\pi_{ijk}^{12}, \Phi_{ij}^{2})}{\longrightarrow} F(\pi_{ijk}^{12}) \circ F(\pi_{ij}^{1}) X_{i}$$

$$\frac{Y_{\pi_{ijk}^{12}, \pi_{ij}^{1}}^{1} X_{i}}{\longrightarrow} F(\pi_{ij}^{1} \circ \pi_{ijk}^{12}) X_{i}$$

$$\frac{Y_{\pi_{ijk}^{13}, \pi_{ik}^{1}}^{1} X_{i}}{\longrightarrow} F(\pi_{ik}^{13} \circ \pi_{ijk}^{13}) X_{i}$$

$$\frac{Y_{\pi_{ijk}^{13}, \pi_{ik}^{1}}^{1} X_{i}}{\longrightarrow} F(\pi_{ijk}^{13}) \circ F(\pi_{ik}^{1}) X_{i}.$$

ST-4: DEFINITION If

are sets of descent data on $\{\textbf{g}_{\textbf{i}} : \textbf{B}_{\textbf{i}} \rightarrow \textbf{B}\} \in \kappa_{\textbf{B'}}$ then a morphism

$$(\{x_{i}\}, \{\phi_{ij}\}) \rightarrow (\{x_{i}'\}, \{\phi_{ij}'\})$$

is a collection of arrows $\xi_{\bf i}\!:\! X_{\bf i}$ \rightarrow $X_{\bf i}^!$ in FB $_{\bf i}$ such that the diagram

$$F(\pi_{ij}^{2})X_{j} \xrightarrow{\phi_{ij}} F(\pi_{ij}^{1})X_{i}$$

$$F(\pi_{ij}^{2})\xi_{j} \downarrow \qquad \qquad \downarrow^{F(\pi_{ij}^{1})\xi_{i}}$$

$$F(\pi_{ij}^{2})X_{j}^{!} \xrightarrow{\phi_{ij}^{!}} F(\pi_{ij}^{1})X_{i}^{!}$$

commutes in FB;

 $\underline{\text{ST-5}}$: NOTATION Given $\{g_i:B_i \to B\} \in \kappa_B$, there is a category

$$F(\{g_i:B_i \rightarrow B\})$$

whose objects are the sets of descent data and whose morphisms are as above.

ST-6: LEMMA The assignment

$$FB \rightarrow F(\{g_i:B_i \rightarrow B\})$$

that sends $X \in FB$ to

$$(\{F(g_i)X\}, \{\phi_{ij}\}),$$

where

$$\phi_{ij} = (\gamma_{ij}^F, x)^{-1} \circ (\gamma_{ij}^F, x),$$

is a functor.

ST-7: DEFINITION Suppose given B and κ -- then a pseudo functor $F:\underline{B}^{OP} \to 2-\mathfrak{CAC}$ is said to be a stack if for all $B \in Ob$ B and all $\{g_{\mathbf{i}}:B_{\mathbf{i}} \to B\} \in \kappa_{B}$, the functor

$$FB \longrightarrow F(\{g_i:B_i \rightarrow B\})$$

is an equivalence of categories.

ST-8: REMARK Consider the setup of 18.12 -- then $F:\underline{C}^{OP} \to \underline{SET}$ is a sheaf iff it is a stack.

[Note: As usual, <u>SET</u> is viewed as a sub-2-category of 2-CAT whose only 2-cells are identities.]

ST-9: EXAMPLE The pseudo functor

$$TOP^{OP} \rightarrow 2-CAC$$

that sends X to TOP/X is a stack in the open subset coverage.

ST-10: EXAMPLE The pseudo functor

that sends X to QCO(X) is a stack in the fpqc coverage (hence in the Zariski coverage, the étale coverage, the smooth coverage, and the fppf coverage).

ST:11: EXAMPLE Given a topological group G, the pseudo functor

that sends B to $\frac{PRIN}{B,G}$ is a stack in the open subset coverage.